

It' as easy as one, two, three ...

In the past two weeks we have been looking at how we use the positive integers to count things, order things, prove things, and some things we might not realize - basically, it appears that there is NO point to this preliminary work that could possibly be focused into a reasonable length, cogent paper.

Does that reflect your experience? Do you have any idea of what we have accomplished, if anything?

1 Axiomatic construction of the real numbers

This is background material not discussed in class.

We build up the real numbers through a sequence of steps. This can be done in a way that only relays on the axiomatic development of the positive integers.

The first step is to construct the positive integers. That sounds weird because we tend to simply accept that they are “there”; that is, we tend to think of them as somehow existing and waiting to be discovered or revealed (taught) to us.

Well, that's not logically sound.

Here's what we do. We take an axiomatic approach; which is, essentially, a run around the question of whether or not the integers exist or are a construct. Recall that axioms, as encountered in geometry, are an expression of “obvious” properties - they can't be proven, they are the foundation for everything we can prove in the system from that starting point.

So, for the integers, we have two axioms that define the set of positive integers:

1. there exists an integer 1,
2. for every integer there is a successor. this is often expressed as a function, called the successor function; for an integer n , there is a next integer $\text{succ}(n) = n + 1$.

In plain language this says (1) I can start counting (at 1), and (2) if I can count up to n , then there is a next number $n + 1$.

Comment: We use the positive integers in two different ways.

1. Integers are used to count distinct elements of a set, for instance, there were 19 students present on Tuesday October 23rd. In this case, we are using the positive integers as *cardinal numbers*, they relate the size or cardinality of a set.
2. Integers are also used to order things, 1st, 2nd, 3rd, *etc.* In this case, the integers are being used as *ordinal numbers*.

Notice that these two concepts are cognitively quite distinct. This is reflected in our language. When we count, we start *one, two, three, ...*, and when we order things, we start *first, second, third,* Some people who study the history of mathematics argue that these uses develop independently of one another, and that some cultures develop ordinals first and that some cultures develop cardinals first. *Why would a culture find ordinals useful?* (Think ceremonies, hierarchy, etc.)

I now digress to outline the construction of the rest of the real numbers.

1. We then add the number 0, which (a) denotes the number of elements in the empty set, and (b) is an additive identity for the integers.
2. Next come the negative integers, which are defined as additive inverses. We need not associate a negative number with a real phenomena to do this: *negative* n is simply a number with the property that (*negative* n) + $n = 0$.
3. Then we construct the rational numbers. Note that this takes a little care.
 - (a) Take all pairs (a, b) , where a is an integer (positive, 0 or negative), and b is a positive integer. So, the pair (a, b) corresponds to the fraction a/b .
 - (b) Then impose an equivalence relation \sim on these pairs. The *fractions* (a, b) and (r, s) are equivalent if and only if $as = rb$. (Oh! This is just the concept of equivalent fractions.)
 - (c) Note then that a rational number is not simply a fraction, but an equivalence class of infinitely many fractions.

4. Next the real numbers are constructed. Note that given a unit length, one can actually construct another length, such as $\sqrt{2}$, which is not a rational length.

There are several ways to construct the real numbers from the rationals.

- (a) The most commonly seen method is axiomatic. This is the approach taken in your MATH 430 book. The existence of the real numbers is assured by the *least upper bound property* which states that every non-empty set of real numbers has a least upper bound. One can define any real, non-rational numbers by applying this property to a suitable set of rational numbers. For instance, let S be the set of all rational numbers x which have the property that $x^2 \leq 2$, then $\sqrt{2}$ is the *lub* of S .
- (b) However, one can construct the real numbers with out introducing new axioms. There are two common ways to do this:
- i. Let \mathcal{S} be the collection of all Cauchy sequences of rational numbers, and define an equivalence relation \equiv by $\{a_i\}_{i=1}^{\infty} \equiv \{b_i\}_{i=1}^{\infty}$ if and only if $\{a_i - b_i\}_{i=1}^{\infty}$ converges to 0. In this view, real numbers are equivalence classes of Cauchy sequences. (Kind of hard to get your head around isn't it?)
 - ii. The second method is to use *Dedekind cuts*. Intuitively, this means that if I take a copy of the real line every point on that line corresponds to a real number - called a cut because cutting the line at that point determines the point. Constructively, this corresponds to dividing the rationals into two sets A and B , where for any $a \in A$ and $b \in B$, $a < b$.
To construct $\sqrt{2}$ as a Dedekind cut, we let
 $A =$ the set of all rational x such that $x^2 \leq 2$, and
 $B =$ the set of all rational x such that $x^2 > 2$.

So, what's the point? It is the fact that starting with the assumption that we can build the positive integers, we can then construct the entire set of real numbers from that set.

2 The ways that we use the integers

We have seen several uses for the integers, other than counting. The two primary uses for the positive integers were discussed above - cardinals and ordinals.

1. Mathematical Induction. Let $S(n)$ be a logical statement using the integer n as a variable (the proper name for such a thing is a propositional function). For example,

$$S(n) : 1 + 2 + \cdots + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

is such a statement. It claims to be true for any integer positive n .

A proof by mathematical induction (weak form) mirrors the axiomatic construction of the positive integers: we first prove that $S(1)$ is true, and then we prove that if $S(n)$ is true, then $S(n+1)$ is true. The logical connection is that as long as we believe that we can continue to count, without bound, then the statement is true because we reduced the problem at hand to the positive integers.

2. Recursion. Recursion is similar to induction, in that it relies on our ability to count. It's different from induction in that induction is used to prove things, while recursion is used to define, or refine, a mathematical concept. For example:
 - (a) Newton's method is a recursive process that refines (improves) an answer. It doesn't always work, but when it does it produces better and better approximates to roots of an equation. We (i) define (guess) an initial value x_1 that we think is close to a root of $f(x)$, then repeatedly (recursively), we define for $i \geq 1$,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

- (b) In class we discussed procedures for effectively producing all r -combinations of an n element set. This was a more complicated thing to explain. We saw the pattern, but it was hard to write

down. That was because the recursion was nested. We continued in a certain way until we ran out “last” numbers, then we went back and changed the next to the last number, etc. Note that there would be r nested steps for this process.

- (c) Sometimes recursion depends on more than the previous answer. This is exhibited in the definition of the Fibonacci sequence, where $f_1 = f_2 = 1$, and for $i > 2$, $f_i = f_{i-1} + f_{i-2}$. This reliance on more than the previous term is analogous to the strong form of mathematical induction.

3. Descent or exhaustion.

Note that examples 1 and 3 of the recursion discussion above involved processes which do not necessarily ever stop. When using Newton’s method to approximate $\sqrt{2}$ we can carry on the process, starting at $x_1 = 1$ forever, and each step gives us a more accurate approximation of the root. In Example 3, there is no last Fibonacci number - the process can be carried on forever.

However, in example 2 the process will end. We depend on integers to bound the number of steps we have to take in many situations.

For example, we considered the Euclidean algorithm for finding the greatest common divisor of two integers. Notice that at each stage of the Euclidean Algorithm the remainder is non-negative and smaller than the previous remainder. This insures that the process will end.

Another example is row reduction of a matrix. After we pivot around the 1,1 entry in a matrix, we then move onto the second row and none of our decisions will depend on the first row - the process ends because we run out of (exhaust) the rows.

Many processes that we encounter end for precisely this reason. At each step we either explicitly or implicitly associate a positive integer with the step, the integer is reduced upon the completion of the step, and thus the process must end because we’ll run out of positive integers.