Linear functions, relations and models

A linear relation is any equation involoving sums of multiples of two variables: we don't have powers or roots of the variables, and we don't divide by them either.

Any linear equation can be put in the *general form*

$$Ax + By + C = 0$$

where A, B and C are constants. But linear equations don't always take that form. Each of the following is also a linear equation.

1. $y = 4x - 3$ has general form $-4x + y + 3 = 0$.
2. $3x - 7 = y$ has general form $3x - y - 7 = 0$.
3. Not all linear equations have to use the variables x and y. $C = 4.3P + 17$ is also a linear equation with general form $C - 4.3P - 17 = 0$.
4. A linear equation in x and y may not have one of the variables in it: the equation $x = 6$ has the general form $1x + 0y = 6$.

In this section we will review the basic aspects of lines - how to find their equations, how to graph them, how they may be related, etc. Additionally, we will discuss how linear equations are used. In particular, the development and use of an equation to investigate an application is called a **model**.

We commonly encounter lines in one of two forms

- $y = mx + b$: This computes a related (dependent) value, y, from a known (independent) value x - it is a function.
- $Ax + By = C$: This equation expresses how we view combinations of things, x *of* A *and* y *of* B *adds up to* C. This is a linear relation.

Examples are commonly encountered.

Example:

1. If Dave gets $185 per month from his parents to help with school, and he works a job that pays $ 10.86 per hour, then how much does money does Dave get each month?

   We should realize that the answer (total money) depends on the number of hours he works, and we aren't given that - it is unknown and variable. Let x stand for the number of hours worked in a month, then Dave makes

   $$E = 10.86x + 185$$

   in a month when he works x hours.

In this case, we see that the Earnings are related to the number of hours worked - in fact, the earnings are dependent upon the hours worked and we can't figure out the earnings without knowing the value of x.

*This is a **function** (or formula) which computes E (the dependent variable) once the numbers of hours (the independent variable) is known.*

If $x = 0$ (he didn't work at all), then $E = 185$.

If $x = 100$, then he earned

$$E = 10.86(100) + 185 = 1,271$$

dollars.

It is easy to get into the habit of always using x and y as our variables, but here we used E because it represents Earnings. We might have chosen h to stand for hours too. That would make it easier to remember what the variables stand for throughout a longer discussion.

Note that models using the form $y = mx + b$ have the following features: there is a quantity, b, that is part of any calculation, and the rest of the calculation is completed by taking a multiple of x (the varying amount).

2. **Linear Combinations:** Denise bought 180 pounds of metal for her sculpture projects. Some of it was bronze, and some was aluminum.

   If x is the amount of bronze, and y is the amount of aluminum, then it is clear that $x + y = 180$ pounds. That expresses the fact that the combined weight of the two types of metals is 180 pounds.

   Alternatively, suppose that the aluminum cost her \$0.93 per pound, and the bronze cost \$2.13 per pound. Then if she buys x pounds of aluminum and y pounds of bronze, her total cost will be $1.23x+0.93y$. If we knew that she spent total of \$127.50, then we would have the linear equation

   $$2.13x + 0.93y = 127.5.$$

   In this case, the form of the equation, $Ax + By = C$, reflects our thinking about "combinations" of things. We could solve the equation $2.13x + 0.93y = 127.5.$ for y - but why? (It would tell us the number of pounds of bronze she bought if you knew the number of pounds of aluminum she bought.)

Example: A linear equation may model many different situations. The equation

$$f(x)0.18x + 38.85$$

could model any of the following:

1. The total bill for electrical service where the base fee is \$38.85 and \$0.18 per Kilowatt Hour (KWH) is charged for x KWH.

2. The cost to produce x embossed business cards, where \$38.85 is the fee for setup, and \$0.18 is charged for each card.

Linear Functions of One Variable

A function of the form

$$y = f(x) = mx + b$$

is called a **linear function**: there is *one independent variable*, x, and one dependent (or related) variable y. In the phrase "linear functions of one variable" - the one variable referred to is the independent variable x. The word "function" indicates that we will use the formula (function) to compute a related value, y, from ONE known value, x.

Recall that we use a "name", that is **functional notation** sometimes. The function $f(x) = 2x + 12$ is a linear function with the name *f of x*. The expression $y = 2x + 12$ expresses the dependent variable y by the same formula as $f(x)$. Functional notation is preferred when we are focusing on the process of finding the related values, or if we have several functions. The notation using the variable y, as in $y = 2x + 12$, is intended to emphasize the geometric relationship that exists between the related values. This class of functions is called *linear* because their graphs are lines.

Example: In the discussion of Dave's monthly income earlier, we found that Dave earned

$$E = f(h) = 10.13h + 185$$

per month, where h is the number of hours worked in the month. Above we used x where we have used h here. Once we know a value for h we can find out his earnings.

Example: Consider a sandwich maker who can make a sandwich for \$3.20 and sell it for \$5.25. It cost him \$35 just to operate his cart per day. Then
his cost for making x sandwiches per day is $C(x) = 3.2x + 35$,
and his revenue (money taken in) is $R(x) = 5.25x$.

Our choice of variables here reflects what they represent. Naming the functions C and R makes it easy to tell which is which. Now we could also compute the profit $P(x)$ form selling x sandwiches:

$$P(x) = R(x) - C(x) = 5.25x - (3.2x + 35) = 2.05x - 35.$$

We now have three values related to x, the number of sandwiches sold, the Cost, Revenue and Profit, and we can tell the functions apart because we are using the functional notation.

When we construct linear equations that are useful in applications like the ones above, we are constructing *mathematical models*. An application that has a *linear model* of the form $f(x) = mx + b$ has two general features:

- a fixed quantity that is always part of the calculation (like the $185 per month Dave got, or the $35 fixed costs for running the sandwich cart).
- an amount that varies as a multiple of a known related quantity (the number of hours worked by Dave, or the number of sandwiches made).

Linear Functions of 2 or More Variables

A linear function in two variables has the form

$$F = Ax + By + C.$$

If we wanted to emphasize the variables, we might write $F(x, y)$. This type of equation generalizes the last concept, but it is not an uncommon model for most people.

> **Example:** Suppose that Dave gets a second job. He now gets $185 per month from his parents, works one job for $10.86 an hour, and suppose he gets a new job for $9.90 per hour. Then his income, E, per month is
>
> $$E = 10.86x + 9.9y + 185$$
>
> where x is the number of hours worked at the $10.86 per hour job, and y the number of hours at the other job.

The more you consider this type of equation, the more natural it appears.

> **Example:** Suppose that Maya has some money invested that earns 8% interest per year, and some invested that earns 9.5% interest per year.
>
> We don't know either value, so we have two unknowns. Let's say x is the amount invested at 8%, and y at 9.5%.
>
> There are two equations involved here.
>
> $$\text{Total invested} = x + y$$
>
> and the total intest earned in a year:
>
> $$\text{Earnings} = 0.08x + 0.095y.$$

It should be clear that there could be situtations where more than two variables are called for. For example, if Dave had three jobs, or Maya had investments at four institutions.

Exercises:

1. If Don has a cell phone plan that costs him 7 cents per minute and $18.90 per month for activation. Give a function $y = f(x)$ that computes his bill for a month if he uses his phone for x minutes per month.
2. John has a vine that is 12 feet long and he has found that it grows 1/2 foot per week. Find a function $g(t)$ that computes the length of the vine after t weeks.

3. Give an example of an application that is modeled by the equation $f(x) = 2.79x + 18.95$.

4. Suppose that Jean has money inveted at iBank and uBank. She gets %3 per year on her money at iBank and 3.5% per year on the money in uBank.

   (a) Assign variables for the amount in each bank.

   (b) If she has a total of $8000 invested, then write an equation that expresses this.

   (c) Give an equation for P, the amount of interest earned in a year.

5. Suppose that John has x dollars and Dawn has y dollars. If Dawn has twice as much money as John, then how are x and y related? (Careful!)

6. If a university advertises that they have 1 professor for every 18 students, then how are P (the no. of professors) related to S (the number of students)?

7. A club has $385 in the bank. They are selling cookies and candles to raise money. The cookies cost them $1.18 per box and they sell them for $3.00, and the candles cost $3.11 and are sold for $6.00. Let x be the number of boxes of cookies sold, and y the number candles sold.

   (a) Write an equation that computes R, the amount they collected from selling cookies and candles.

   (b) Find an equation for the cost C of the cookies and candles.

   (c) The profit is $R - C$, find an equation for the profit P.

   (d) If they deposit all of the profit into the bank, then what will their new balance be?

   (e) If they sold 118 boxes of cookies and 49 candles, then what will their new bank balance be?