Lecture notes: 10.4: Exponential Growth and Decay

We operate under the assumption that growth rate is proportional to population size. $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is the proportionality constant. This is called the law of natural growth when $k > 0$ and the law of natural decay when $k < 0$.
consider $\int \frac{dy}{y} = \int kdt$

$$\Longrightarrow \ln|y| = dt + c \Longrightarrow |y| = e^{kt+c} = e^c e^{kt} \Longrightarrow y = Ae^{kt}$$

Where $A = +(-)e^c$ is an arbitrary constant. Now $y(0) = Ae^{k0} = A$ so A is the initial value of the function. So given $\frac{dy}{dt} = ky, y(0) = y_0 \Longrightarrow y(t) = y_0 e^{kt}$

Find $f(t)$ given that $f'(t) = 2f(t)$ for every t and $f(0) = \sqrt{2}$

$$f'(t) = 2f(t) \Longrightarrow f(t) = ce^{2t} \Longrightarrow f(t) = \sqrt{2}e^{2t}$$

Population Growth: $\frac{dP}{dt} = kP \Longrightarrow \frac{1}{P}\frac{dP}{dt} = k$ where $\frac{1}{P}\frac{dP}{dt}$ is the growth rate divided by the population size and is called the relative growth rate.

Examples:

1. Population tends to grow with time at a rate roughly proportional to the population at present. According to the Bureau of the Census, the population of the United States in 1970 was 203 million and in 1980, 227 million. Use this information to estimate the year in which the population of the United States may be expected to be 5 billion.

   Solution: Let $P(t)$ be the population in millions of the United States. t years after 1970,

$$P'(t) = kP(t) \Longrightarrow P(t) = Ae^{kt}$$

$$P(0) = 203 \Longrightarrow P(t) = 203e^{kt}$$

$$P(10) = 227 \Longrightarrow 227 = 203e^{10k} \Longrightarrow e^{10k} = \frac{227}{203} \Longrightarrow e^k = \left(\frac{227}{203}\right)^{1/10}$$

$$\Longrightarrow P(t) = 203\left(\frac{227}{203}\right)^{t/10}$$

$$\Longrightarrow 203\left(\frac{227}{203}\right)^{t/10} = 5000 \Longrightarrow \ln 203 + \frac{t}{10}\ln\left(\frac{227}{203}\right) = \ln 5000$$

$$\Longrightarrow t = \frac{10\ln(5000/203)}{\ln(227/203)}$$

Which is approximately 287, so the US population will be 5 billion in the year 2257.

2. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria.

   (a) Find an expression for the number of bacteria after t hours.
   Solution: $A(t) = A(0)e^{kt} = 500e^{kt} \Longrightarrow A(3) = 500e^{3k} = 8000 \Longrightarrow e^{3k} = 16 \Longrightarrow 3k = \ln 16 \Longrightarrow k = \dfrac{\ln 16}{3} \Longrightarrow A(t) = 500e^{(\ln 16)t/3} = 500(16^{t/3})$

   (b) Find the number of bacteria after 4 hours.
   Solution: $A(4) = 500(16^{4/3})$ which is approximately 20,159

   (c) Find the rate of growth after 4 hours.
   Solution: $\dfrac{dA}{dt} = kA \Longrightarrow A'(4) = kA(4) = \dfrac{1}{3}\ln 16(500(16^{4/3})$

   (d) When will the population reach 30,000?
   Solution: $A(t) = 500(16^{t/3}) = 30000 \Longrightarrow 16^{t/3} = 60 \Longrightarrow \dfrac{1}{3}t\ln 16 = \ln 60 \Longrightarrow t = \dfrac{3\ln 60}{\ln 16}$ which is approximately 4.4 hours.

Radioactive Decay: Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate has been found to be $-\dfrac{1}{m}\dfrac{dm}{dt} \Longrightarrow \dfrac{dm}{dt} = km$ where k is a negative constant. So radioactive substances decay at a rate proportional to the remaining mass $\Longrightarrow m(t) = m_0e^{kt}$.

Examples:

1. Today we have m_0 grams of a radioactive substance. Given that 1/3 of the substance decays every 5 years, how much will be left t years from today?

   Solution: We know $A'(t) = kA(t) \Longrightarrow A(t) = ce^{kt}$ where $c = A_0 \Longrightarrow A(t) = A_0e^{kt}$. At the end of 5 years, 1/3 of A_0 will have decayed so 2/3 of A_0 is left.

   $$\Longrightarrow A(5) = \frac{2}{3}A_0 \Longrightarrow A(5) = A_0e^{5k} = \frac{2}{3}A_0 \Longrightarrow e^{5k} = \frac{2}{3} \Longrightarrow e^k = \left(\frac{2}{3}\right)^{1/5}$$

   $$\Longrightarrow A(t) = A_0\left(\frac{2}{3}\right)^{t/5}$$

2. What is the half-life of the above substance (how long does it take for 1/2 of the substance to decay)?

   Solution: $A_0\left(\frac{2}{3}\right)^{t/5} = \frac{1}{2}A_0 \Longrightarrow \left(\frac{2}{3}\right)^{t/5} = \frac{1}{2} \Longrightarrow \left(\frac{2}{3}\right)^{t} = \left(\frac{1}{2}\right)^{5} \Longrightarrow$ $t\ln(2/3) = 5\ln(1/2) \Longrightarrow t(\ln 2 - \ln 3) = -5\ln 2 \Longrightarrow t = \dfrac{5\ln 2}{\ln 3 - \ln 2}$. So the half life of the substance is approximately 8.55 years.

3. 2 years ago there were 5 grams of a radioactive substance. Now there are 4 grams. How much will remain 3 years from now?

   Solution: take 2 years ago as $t = 0$. So we have $A(t) = A_0e^{kt}, A_0 = 5, A(2) = 4$.

   $$\Longrightarrow 4 = 5e^{2k} \Longrightarrow \frac{4}{5} = e^{2k} \Longrightarrow e^k = \left(\frac{4}{5}\right)^{t/2}$$

   $$\Longrightarrow A(t) = 5\left(\frac{4}{5}\right)^{t/2} \Longrightarrow A(5) = 5\left(\frac{4}{5}\right)^{5/2}$$

   So approximately 2.86 grams will remain 3 years from now.

Continuously Compounded Interest: Consider a given amount of money invested at an interest rate r. If the accumulated interest is credited once a year it is said to be compounded annually. If it is credited twice a year it is said to be compounded semi-annually, etc. Interest can be credited every day, every hour, every second, every half-second, and so on. This is called continuously compounded interest.

$$A(t) = A_0e^{rt}$$

where t is measured in years, $A(t)$ is the principal in dollars at time t, $A_0 = A(0)$ is the initial investment, and r is the annual interest rate. r is called the nominal interest rate. Compounding makes the effective rate higher.

Examples:

1. Find the amount of interest earned by \$100 compounded continuously at 6% for 5 years.

   Solution: $A(5) = 100e^{(.06)5} = 100e^{.3}$ which is approximately 134.99 so the interest earned is approximately \$34.99.

2. A sum of money is earning interest at the rate of 10% compounded continuously. What is the effective interest rate?

   Solution: at the end of one year each dollar grows to $e^{.1}$ which is approximately 1.105 so the interested earned is 10.5 cents and so the effective interest rate is 10.5%

3. How long does it take a sum of money to double at 5% compounded continuously?

   Solution: $A(t) = A_0e^{(.05)t} \Longrightarrow 2A_0 = A_0e^{(.05)t} \Longrightarrow 2 = e^{(.05)t} \Longrightarrow \ln 2 = .05t = \frac{1}{20}t \Longrightarrow t = 20 \ln 2$. So it takes approximately 13.86 years to double.

Homework: 4,8,18