Lecture notes 7.7: Indeterminate Forms and L'Hospital's Rule

Recall:
Defintion: We write $\lim\limits_{x\to a} f(x) = k$ if we can make the values of $f(x)$ arbitrarily close to k by taking x sufficiently close to a, but not equal to a.
If $\lim\limits_{x\to a} f(x) = k$ and $\lim\limits_{x\to a} g(x) = m$ then

1. $\lim\limits_{x\to a}[f(x) + g(x)] = k + m$
2. $\lim\limits_{x\to a}[\alpha f(x)] = \alpha k$ for each real α
3. $\lim\limits_{x\to a}[f(x)g(x)] = km$
4. If $\lim\limits_{x\to a} g(x) = m$ with $m \neq 0$ then $\lim\limits_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$

This section deals with indeterminate forms of limits. What are the indeterminate forms?

1. $f(x) \to 0, g(x) \to 0 \Longrightarrow \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$
2. $f(x) \to \infty, g(x) \to \infty \Longrightarrow \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\infty}{\infty}$
3. $f(x) \to 0, g(x) \to \infty \Longrightarrow \lim\limits_{x\to a} f(x)g(x) = 0\infty$
4. $f(x) \to \infty, g(x) \to \infty \Longrightarrow \lim\limits_{x\to a}[f(x) - g(x)] = \infty - \infty$
5. $f(x) \to 0, g(x) \to 0 \Longrightarrow \lim\limits_{x\to a} f(x)^{g(x)} = 0^0$
6. $f(x) \to \infty, g(x) \to 0 \Longrightarrow \lim\limits_{x\to a} f(x)^{g(x)} = \infty^0$
7. $f(x) \to 1, g(x) \to \infty \Longrightarrow \lim\limits_{x\to a} f(x)^{g(x)} = 1^\infty$

For (1) we consider the special case when $f(a) = g(a) = 0$:

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

The general version of l'Hospital's Rule for the indeterminate form $\frac{0}{0}$ is somewhat more difficult.

Suppose that the function f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for every $x \in (a, b)$. Then there is a number $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

This is a Cauchy's Mean Value Theorem.

We assume $\lim_{x \to a} f(x) = 0, \lim_{x \to a} g(x) = 0$. Let $L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

We must show that $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Define

$$F(x) = \begin{cases} f(x) & x \neq a \\ 0 & x = a \end{cases}$$

$$G(x) = \begin{cases} g(x) & x \neq a \\ 0 & x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I | x \neq a\}$ and $\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$. Likewise, G is continuous on I. Let $x \in I$, and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G'(x) \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem ther is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F(x)}{G(x)}$$

$(F(a) = G(a) = 0)$.

let $x \to a^+, y \to a^+$ since $a < y < x$, so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{y \to a^+} \frac{F'(y)}{G'(y)} = \lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore $\lim_{x \to a} \frac{f(x)}{g(x)} = L$. This proves L'Hospital's Rule for the case where a is finite.

L'hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possible at a). Suppose that:

$$\lim_{x\to a} f(x) = 0, \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \infty, \qquad \lim_{x\to a} g(x) = \infty$$

in other words, we have an indeterminate form of (1) or (2), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exits (or is ∞ or $-\infty$).

Remarks:

1. LR says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using LR.

2. LR is also valid for one-sided limits and for limits at ∞ or $-\infty$.

Examples:

1. Find $\lim\limits_{x\to \pi/2} \dfrac{\cos x}{\pi - 2x}$

   as $x \to \pi/2, \cos x \to 0, \pi - 2x \to 0$ so the conditions are met.
   $f(x) = \cos x \Longrightarrow f'(x) = -\sin x$
   $g(x) = \pi - 2x \Longrightarrow g'(x) = -2$
   $\Longrightarrow \dfrac{f'(x)}{g'(x)} = \dfrac{-\sin x}{-2} = \dfrac{\sin x}{2} \to \dfrac{1}{2}$ as $x \to \pi/2$
   $\lim\limits_{x\to \pi/2} \dfrac{\cos x}{\pi - 2x} = \lim\limits_{x\to \pi/2} \dfrac{\sin x}{2} = \dfrac{1}{2}$

2. Find $\lim\limits_{x\to 0^+} \dfrac{x}{\sin\sqrt{x}}$

   as $x \to 0^+$ both numerator and denominator tend to 0
   $\dfrac{f'(x)}{g'(x)} = \dfrac{1}{(\cos\sqrt{x})(1/2\sqrt{x})} = \dfrac{2\sqrt{x}}{\cos\sqrt{x}} \to 0$ as $x \to 0^+$
   LR $\Longrightarrow \dfrac{x}{\sin\sqrt{x}} \to 0$ as $x \to 0^+$

3. Find $\lim\limits_{x\to 0^+} \dfrac{e^x - x - 1}{x^2}$

   as $x \to 0$, both numerator and denominator tend to 0.
   $\dfrac{f'(x)}{g'(x)} = \dfrac{e^x - 1}{2x}$. Since both numerator and denominator tend to 0, we differentiate again:
   $\dfrac{f''(x)}{g''(x)} = \dfrac{e^x}{2} \to \dfrac{1}{2} \Longrightarrow \dfrac{e^x - 1}{2x} \to \dfrac{1}{2} \Longrightarrow \dfrac{e^x - x - 1}{x^2} \to \dfrac{1}{2}$

4. Show that $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$

   both the numerator and denominator tend to ∞ with x.
   LR $\Longrightarrow \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$

5. Find $\lim\limits_{x\to 0^+} x \ln x$

   We rewrite this as $\lim\limits_{x\to 0^+} x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\dfrac{1}{x}}$ so that we satisfy the conditions for LR.
   $\Longrightarrow \lim\limits_{x\to 0^+} x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{1/x} = \lim\limits_{x\to 0^+} \dfrac{1/x}{-1/x^2} = \lim\limits_{x\to 0^+} (-x) = 0$
   The way we handle 0∞ is to rewrite it in the form of $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

6. find $\lim\limits_{x\to(\pi/2)^-} (\sec x - \tan x)$ which has the form $\infty - \infty$

   $\lim\limits_{x\to(\pi/2)^-} (\sec x - \tan x) = \lim\limits_{x\to(\pi/2)^-} \left(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x} \right) = \lim\limits_{x\to(\pi/2)^-} \dfrac{1 - \sin x}{\cos x} = \lim\limits_{x\to(\pi/2)^-} \dfrac{-\cos x}{-\sin x} = 0$
   $1 - \sin x \to 0, \cos x \to 0$ as $x \to (\pi/2)^-$

7. calculate $\lim\limits_{x\to 0^+} (1 + \sin 4x)^{\cot x}$

   as $x \to 0^+, 1 + \sin 4x \to 1, \cot x \to \infty$
   let $y = (1 + \sin 4x)^{\cot x}$
   $\Longrightarrow \ln y = \cot x \ln(1 + \sin 4x) = \dfrac{\ln(1 + \sin 4x)}{\tan x}$
   LR $\Longrightarrow \lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln(1 + \sin 4x)}{\tan x} = \lim\limits_{x\to 0^+} \dfrac{\dfrac{4\cos 4x}{1 + \sin 4x}}{(\sec x)^2} = 4$

so far we have computed the limit of $\ln y$, but what we want is the limit of y. To do this we use the fact that $y = e^{\ln y}$
$\lim\limits_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim\limits_{x \to 0^+} y = \lim\limits_{x \to 0^+} e^{\ln y} = e^4$

8. find $\lim\limits_{x \to 0^+} x^x$

   $x^x = e^{x \ln x} \Longrightarrow \lim\limits_{x \to 0^+} x^x = \lim\limits_{x \to 0^+} e^{x \ln x} = e^0 = 1$ (from (5))