Lecture notes 8.3: Trigonometric Substitution

Integrals involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$ require trigonometric substitution to calculate. We use inverse substitutions to solve such integrals: a substitution of the form $x = g(t)$ to get $\int f(x)dx = \int f(g(t))g'(t)dt$. So if we have $\int \sqrt{a^2-x^2}dx$, we substitute $x = a\sin\theta$ to transform $\sqrt{a^2-x^2} \Longrightarrow \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$. We need the inverse substitution to define a one-to-one function, so we have to restrict θ to the interval $[-\pi/2, \pi/2]$.

1. For $\sqrt{a^2-x^2}$ set $a\sin u = x$
2. For $\sqrt{a^2+x^2}$ set $a\tan u = x$
3. For $\sqrt{x^2-a^2}$ set $a\sec u = x$

In each case take $a > 0$.

Examples:

1. Find $\int \frac{dx}{(a^2-x^2)^{3/2}}$

   Solution: Set $a\tan u = x, a\sec^2 udu = dx$
   $$\Longrightarrow \int \frac{dx}{(a^2-x^2)^{3/2}} = \int \frac{a\cos u}{(a^2-a^2\sin^2 u)^{3/2}}du$$
   $$= \frac{1}{a^2}\int \frac{\cos u}{\cos^3 u}du = \frac{1}{a^2}\int \sec^2 udu$$
   $$= \frac{1}{a^2}\tan u + c = \frac{x}{a^2\sqrt{a^2-x^2}} + c$$

2. Find $\int \sqrt{a^2+x^2}dx$

   Solution: Set $a\tan u = x, a\sec^2 udu = dx$
   $$\Longrightarrow \int \sqrt{a^2+x^2}dx = \int \sqrt{a^2+a^2\tan^2 u}a\sec^2 udu$$
   $$= a^2\int \sqrt{1+\tan^2 u}\sec^2 udu = a^2\int \sec u\sec^2 udu$$
   $$= a^2\int \sec^3 udu = \frac{a^2}{2}(\sec u\tan u + \ln|\sec u + \tan u|) + c$$
   $$= \frac{a^2}{2}\left[\frac{\sqrt{a^2+x^2}}{a}\left(\frac{x}{a}\right) + \ln\left[\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right]\right] + c$$

$= \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln(x+\sqrt{a^2+x^2}) - \frac{1}{2}a^2 \ln a + c$
$= \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln(x+\sqrt{a^2+x^2}) + C$

3. Find $\int \frac{dx}{x\sqrt{4x^2+9}}$.

   Solution: set $3\tan u = 2x, 3\sec^2 u du = 2dx$
   $\int \frac{dx}{x\sqrt{4x^2+9}} = \int \frac{3/2\sec^2 u}{3/2\tan u 3\sec u} du$
   $= \frac{1}{3}\int \frac{\sec u}{\tan u} du = \frac{1}{3}\int \csc u du$
   $= \frac{1}{3}\ln|\csc u - \cot u| + c$
   $= \frac{1}{3}\ln\left[\frac{\sqrt{4x^2+9}-3}{2x}\right] + c$

4. Find $\int \frac{x}{\sqrt{x^2+2x-3}}dx$. (involves completing the sqare)

   Solution: First note that $\int \frac{x}{\sqrt{x^2+2x-3}}dx = \int \frac{x}{\sqrt{(x+1)^2-4}}dx$
   Now set $2\sec u = x+1, 2\sec u \tan u = dx$
   Then $\int \frac{x}{\sqrt{(x+1)^2-4}}dx = \int \frac{(2\sec u - 1)2\sec u \tan u}{2\tan u} du$
   $= \int (2\sec^2 u - \sec u) du$
   $= 2\tan u - \ln|\sec u + \tan u| + c$
   $= \sqrt{x^2+2x-3} - \ln\left[\frac{x+1+\sqrt{x^2+2x-3}}{2}\right] + c$

5. Find $\int \frac{\sqrt{9-x^2}}{x^2}dx$.

   Solution: Set $x = 3\sin u, dx = 3\cos u$
   Then $\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3\cos u}{9\sin^2 u} 3\cos u du$
   $= \int \frac{\cos^2 u}{\sin^2 u} du = \int \cot u^2 du$
   $= \int (\csc^2 u - 1) du = -\cot u - u + c$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$$

6. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   Solution: $y = \frac{b}{a}\sqrt{a^2 - x^2}$ or $y = -\frac{b}{a}\sqrt{a^2 - x^2}$
   Because the ellipse is symmetric with respect to both axes, the total area is 4 times the area in the first quadrant. The part of the ellipse in the first quadrant is given by the function $y = \frac{b}{a}\sqrt{a^2 - x^2}$ for $0 \le x \le a$.
   $\Longrightarrow A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$
   Let $x = a\sin u, dx = a\cos u du$
   $\Longrightarrow \frac{4b}{a}\int_0^{\pi/2} a\cos u a\cos u du$
   $= 4ab\int_0^{\pi/2} \cos^2 u du = 4ab\int_0^{\pi/2} \frac{1}{2}(1 + \cos 2u) du$
   $= 2ab\left[u + \frac{1}{2}\sin 2u\right]_0^{/pi/2}$
   $2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab$