

Lecture notes 8.8: Improper Integrals

In defining a definite integral $\int_a^b f(x)dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity. Now we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case, the integrals are called indefinite integrals. The major application of these is in probability distributions.

Definition of an improper integral of type 1: If $\int_a^b f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

any real number a can be used.

Definition of an improper integral of type 2: If f is continuous on $[a, b)$ and is discontinuous at b , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number).

If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

If f has a discontinuity at c where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
2. If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.