

Lecture notes 12.3: The Integral Test and Estimates of Sums

Because it is not usually easy to find the exact sum of series, we will develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. In some cases, we will be able to find good estimates of sums.

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words,

1. If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent
2. If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

Note: When we use the integral test, it is not necessary to start the series or the integral at $n = 1$.

Recall the example from improper integrals for $\int_1^{\infty} \frac{1}{x^p} dx$. Which gives us the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ which is convergent if $p > 1$ and divergent if $p \leq 1$.

Note: the sum of the series is not necessarily equal to the value of the integral. For example, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\int_1^{\infty} \frac{1}{x^2} dx = 1$

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

$f(x) = \frac{\ln x}{x}$ is positive and continuous for $x > 1$. To determine whether it is decreasing, we compute the derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1, (x > e)$. so f is decreasing when $x > e$ and so we can apply the integral test:

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^t = \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} = \infty$$

thus our series diverges.

Estimating the sum of a series involves the remainder R_n which is the error made when s_n is used as an approximation of the total sum. If $\sum a_n$ converges by the integral test and $R_n = s - s_n$, then

$$\begin{aligned} \int_{n+1}^{\infty} f(x)dx &\leq R_n \leq \int_n^{\infty} f(x)dx \\ \implies s_n + \int_{n+1}^{\infty} f(x)dx &\leq s \leq s_n + \int_n^{\infty} f(x)dx \end{aligned}$$

which gives us a lower and upper bound for s . They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 15 terms.

We start with $\int_n^{\infty} \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2x^2} \right]_n^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$

then we approximate the sum with the first 15 terms $s_{15} = \frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{15^3}$

and our error is $R_{15} \leq \int_{15}^{\infty} \frac{1}{x^3} dx = \frac{1}{(15)^3}$

or we can use $s_{15} + \int_{16}^{\infty} \frac{1}{x^3} dx \leq s \leq s_{15} + \frac{1}{x^3} dx$

$$\implies s_{15} + \frac{1}{2(16)^2} \leq s \leq s_{15} + \frac{1}{2(15)^2}$$

How many terms are required to ensure that the sum is accurate to within .00005?

We need to find a value of n such that $R_n \leq .00005$

$$R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

so we need $\frac{1}{2n^2} < .00005$ and just solve the inequality.