Lecture notes: 6.3: Volumes by Cylindrical Shells

Some problems are very difficult to handle by the method of discs in 6.2. i.e. if we consider the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y, which is not easy. So we introduce a new method, called the **method of cylindrical shells**, that is easier to use in such cases. (show transparency 9). figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi r_2^2 h - \pi_1^2 h = \pi(r_2^2 - r_1^2)h = \pi(r_2 + r_1)(r_2 - r_1)h = 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)$$

If we let $\triangle r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then the formula for the volume of a cylindrical shell becomes

$$V = 2\pi r h \triangle r$$

and it can be remembered as V =(circumference)(height)(thickness). Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geq 0$], $y = 0$, $x = a$, $x = b$, where $b > a \geq 0$ (figure 3). We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\triangle x$. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average radius x_i^*, height $f(x_i^*)$, and thickness $\triangle x$ (figure 4). so by our formula,

$$V_i = (2\pi x_i^*)[f(x_i^*)]\triangle x$$

So following our familiar pattern after Riemann sums we see that

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} (2\pi x_i^*)[f(x_i^*)]\triangle x = \int_a^b 2\pi x f(x) dx$$

where $0 \leq a < b$. Examples:

1. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$

   Solution:

   (a) draw picture

   (b) determine limits of integration

   $$2x^2 - x^3 = 0 \Longrightarrow x^2(2 - x) = 0 \Longrightarrow x = 0, x = 2$$

   (c) set up integral

   $$V = \int_0^2 (2\pi x)(2x^2 - x^3)dx = 2\pi \int_0^2 (2x^3 - x^4)dx$$

   (d) evaluate integral

   $$= 2\pi \left[\frac{1}{2}x^4\frac{1}{5}x^5\right]_0^2 = 2\pi \left(8 - \frac{32}{5}\right) = \frac{16\pi}{5}$$

2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$

   Solution:

   (a) draw picture

   (b) determine limits of integration

   $$x = x^2 \Longrightarrow x^2 - x = 0 \Longrightarrow x(x - 1) = 0 \Longrightarrow x = 0, x = 1$$

   (c) set up integral

   $$V = \int_0^1 (2\pi x)(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$$

   (d) evaluate the integral

   $$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$

   Solution:

(a) draw picture

(b) determine the limits of integration

$$x - x^2 = 0 \Longrightarrow x(1-x) = 0 \Longrightarrow x = 0, x = 1$$

(c) set up the integral

$$V = \int_0^1 2\pi(2-x)(x-x^2)dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)dx$$

(d) evaluate the integral

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$

4. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$, $x = 1$, $x = 2$, about $x = 4$

   Solution:

   (a) draw picture

   (b) determine the limits of integration

   (c) set up integral

$$V = \int_1^2 2\pi(4-x)x^2dx = 2\pi \int_1^2 (4x^2 - x^3)dx$$

   (d) evaluate integral

$$= 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_1^2 = 2\pi \left[\left(\frac{32}{3} - 4\right) - \left(\frac{4}{3} - \frac{1}{4}\right)\right] = \frac{67\pi}{6}$$

5. The integral $\int_0^9 2\pi y^{3/2}dy$ represents a solid - describe it.

   Solution: The solid is obtained by rotating the region bounded by the curve $x = \sqrt{y}$ and the lines $y = 9$ and $x = 0$ about the x-axis

6. $y = 5$, $y = x^2 - 5x + 9$ about $x = -1$

   solution:

(a) draw picture

(b) determine limits of integration

$$x^2-5x+9=5 \Longrightarrow x^2-5x+4=0 \Longrightarrow (x-4)(x-1)=0 \Longrightarrow x=4, x=1$$

(c) set up integral

$$V=\int_1^4 2\pi[x-(-1)][5-(x^2-5x+9)]dx = 2\pi\int_1^4 (x+1)(-x^2+5x-4)dx$$

$$= 2\pi\int_1^4 (-x^3+4x^2+x-4)dx$$

(d) evaluate integral

$$= 2\pi\left[\left(-64+\frac{256}{3}+8-16\right)-\left(-\frac{1}{4}+\frac{4}{3}+\frac{1}{2}-4\right)\right] = 2\pi\left(\frac{63}{4}\right) = \frac{63\pi}{2}$$

Homework: 10, 20, 36, 40