Lecture notes: 7.1: Inverse Functions

Definition: A function f is called a one-to-one function if it never takes on the same value twice. i.e. $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Recall: if a horizontal line intersects the graph of f in more than one point, then it is not one-to-one (horizontal line test)

One-to-one functions are important because they are precisely the functions that posess inverse functions.

Definition: Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Longleftrightarrow f(x) = y$$

for any $y \in B$

so domain of f^{-1} = range of f and range of f^{-1} = domain of f. Remember how to find the domain and range of a function?

Which of the following are one-to-one:

1. $f(x) = x^2$
2. $f(x) = x^3$
3. $f(x) = x^4$
4. $f(x) = x^5$

Caution: Do not mistake the -1 in f^{-1} for an exponent. $f^{-1} \neq \dfrac{1}{f(x)}$ The reciprocal of $f(x)$ is $\dfrac{1}{f(x)} = [f(x)]^{-1}$.

The following are true:

1. $f^{-1}(x) = y \Longleftrightarrow f(y) = x$
2. $f^{-1}(f(x)) = x \quad \forall x \in A$
3. $f(f^{-1}(x)) = x \quad \forall x \in B$

(f inverse undoes what f does and f undoes what f inverse does) How to find the inverse function of a one-to-one function f;

1. write $y = f(x)$
2. solve this equation for x in terms of y (if possible)
3. to express f^{-1} as a function of x, interchange x and y which results in $y = f^{-1}(x)$
4. the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$

sketch the following graphs with their inverses:

1. $f(x) = x^3$
2. $f(x) = x^{1/2}$
3. $f(x) = \sqrt{-1-x}$

find the following inverse functions:

1. $f(x) = x^3 + 2$

   Solution: $y = x^3 + 2 \Longrightarrow y - 2 = x^3 \Longrightarrow x = (y-2)^{1/3} \Longrightarrow$

   $$f^{-1}(x) = (x-2)^{1/3}$$

2. $f(x) = \sqrt{-1-x}$

   Solution: $y = (-1-x)^{1/2} \Longrightarrow y^2 = -1 - x \Longrightarrow x = -1 - y^2 \Longrightarrow$

   $$f^{-1}(x) = -1 - x^2$$

3. $f(x) = 5 - 4x^3$

   Solution: $y = 5 - 4x^3 \Longrightarrow 4x^3 = 5 - y \Longrightarrow x^3 = \dfrac{5-y}{4} \Longrightarrow x = \left(\dfrac{5-y}{4}\right)^{1/3} \Longrightarrow$

   $$f^{-1}(x) = \left(\frac{5-x}{4}\right)^{1/3}$$

4. $f(x) = \dfrac{1+3x}{5-2x}$

   Solution: $y = \dfrac{1+3x}{5-2x} \Longrightarrow y(5-2x) = 1+3x \Longrightarrow 5y-2xy = 1+3x \Longrightarrow$
   $3x + 2xy = 5y - 1 \Longrightarrow x(3+2y) = 5y-1 \Longrightarrow x = \dfrac{5y-1}{3+2y} \Longrightarrow$

$$f^{-1}(x) = \frac{5x-1}{3+2x}$$

The calculus of inverse functions:

1. If f is 1 to 1 and continuous defined on an interval, then its inverse functions f^{-1} is also continuous.

2. if f is 1 to 1 and differentiable with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and

$$g'(a) = \frac{1}{f'(g(a))}$$

use calculus to find $f^{-1}(a)$, find $g = f^{-1}$, find the domain and range of g:

1. $f(x) = x^3, a = 8$

   (a) show its one to one
   (b) $x^3 = 8 \Longrightarrow x = 8^{1/3} \Longrightarrow x = 2 \Longrightarrow g(a) = g(8) = 2$
   (c) $f(x) = x^3 \Longrightarrow f'(x) = 3x^2$
   (d) $g'(8) = \dfrac{1}{f'((g(8))} = \dfrac{1}{f'(2)} = 1/12$
   (e) $y = x^3 \Longrightarrow y^{1/3} = x \Longrightarrow f^{-1}(x) = x^{1/3}$

2. $f(x) = \sqrt{x-2}, a = 2$

   (a) show its one to one
   (b) $(x-2)^{1/2} = 2 \Longrightarrow x - 2 = 4 \Longrightarrow x = 4 \Longrightarrow f(6) = (6-2)^{1/2} = 2 = a \Longrightarrow g(2) = 6$

(c) $f(x) = (x-2)^{1/2} \Longrightarrow f'(x) = \frac{1}{2}(x-2)^{-1/2} \Longrightarrow f'(6) = \frac{1}{2}(6-2)^{-1/2} = 1/4$

(d) $g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(6)} = \frac{1}{1/4} = 4$

(e) $y = (x-2)^{1/2} \Longrightarrow y^2 = x - 2 \Longrightarrow y^2 + 2 = x \Longrightarrow f^{-1}(x) = x^2 + 2$

3. $f(x) = \frac{1}{x-1}, x > 1, a = 2$

   (a) show its one to one

   (b) $\frac{1}{x-1} = 2 \Longrightarrow 1 = 2x - 2 \Longrightarrow 2x = 3 \Longrightarrow x = 3/2 \Longrightarrow g(2) = 3/2$

   (c) $f(x) = \frac{1}{x-1} \Longrightarrow f'(x) = -(x-1)^{-2} \Longrightarrow f'(3/2) = \frac{1}{(3/2-1)^2} = -4$

   (d) $g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(3/2)} = -1/4$

   (e) $y = \frac{1}{x-1} \Longrightarrow 1 = y(x-1) \Longrightarrow 1 = yx - y \Longrightarrow 1 + y = yx \Longrightarrow \frac{1}{y} + 1 = x \Longrightarrow f^{-1}(x) = \frac{1}{x} + 1$

Homework: 10,16,22,40