

Lecture notes: 7.5: Inverse Trigonometric Functions

From 7.1, we know that continuous one-to-one functions have inverses but none of the trig functions are one-to-one so none of them have inverses. What then are the inverse trig functions?

The inverse sine function: The range of the sine function is the closed interval $[-1, 1]$. Although not one-to-one on its full domain, the sine function is one-to-one on the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and on that interval it takes on as a value every number in $[-1, 1]$. Thus, if $x \in [-1, 1]$, there is one and only one number in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ at which the sine function takes on the value x . This number is called the inverse sine of x and is written $\sin^{-1} x$ or $\arcsin x$. The inverse sine function

$$y = \sin^{-1} x \quad x \in [-1, 1]$$

is the inverse of the function

$$y = \sin x \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

The graphs of each function is the reflection of the other about the line $y = x$. Because they are inverses, for all $x \in [-1, 1]$,

$$\sin(\sin^{-1} x) = x$$

and for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\sin^{-1}(\sin x) = x$$

Also, being an inverse of an odd function, sine inverse is also an odd function and so $\sin^{-1}(-x) = -\sin^{-1} x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Calculate if defined:

1. $\sin^{-1}(\sin \frac{\pi}{16})$

since $\frac{\pi}{16}$ is within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we know that $\sin^{-1}(\sin \frac{\pi}{16}) = \frac{\pi}{16}$

2. $\sin^{-1}(\sin \frac{5\pi}{2})$

since $\frac{5\pi}{2}$ is not within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we cannot apply our formula

directly, but $\frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$. Thus

$$\sin^{-1}(\sin \frac{5\pi}{2}) = \sin^{-1}(\sin(\frac{\pi}{2} + 2\pi)) = \sin^{-1}(\sin \frac{\pi}{2}) = \frac{\pi}{2}$$

3. $\sin^{-1}(\sin \frac{9\pi}{5})$

since $\sin \frac{9\pi}{5}$ is not within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we cannot apply our

formula directly, but $\frac{9\pi}{5} = 2\pi - \frac{\pi}{5}$. Thus

$$\sin^{-1}(\sin \frac{9\pi}{5}) = \sin^{-1}(\sin(2\pi - \frac{\pi}{5})) = \sin^{-1}(\sin(-\frac{\pi}{5})) = -\frac{\pi}{5}$$

4. $\sin(\sin^{-1} 2)$

The expression doesn't make sense since 2 is not within the domain of the inverse sine. The inverse sine is defined only on $[-1, 1]$.

Let $y = \sin^{-1} x \implies \sin y = x$, $-\pi/2 \leq y \leq \pi/2$ and we use implicit differentiation to get

$$\cos y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y}$$

$$-\pi/2 \leq y \leq \pi/2 \implies y \geq 0 \implies \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

remember the trig identity $\sin^2 y + \cos^2 y = 1$ and that $x = \sin y$

$$\text{thus we get } \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

Examples

1. Find $\frac{d}{dx}[\sin^{-1}(3x^2)]$

In general, from the chain rule we get $\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}$ and

$$\text{so } \frac{d}{dx}[\sin^{-1}(3x^2)] = \frac{1}{\sqrt{1 - (3x^2)^2}} \frac{d}{dx}(3x^2) = \frac{6x}{\sqrt{1 - 9x^4}}$$

2. Show that for $a > 0$ $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c$

We change variables so that the a^2 becomes 1 and we can use our formula. We set $au = x \implies adu = dx \implies \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{adu}{\sqrt{a^2 - a^2u^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + c = \sin^{-1} \left(\frac{x}{a} \right) + c$

3. Evaluate $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}}$

$$\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1} \left(\frac{x}{2} \right) + c \implies \int_0^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}} = \left[\sin^{-1} \left(\frac{x}{2} \right) \right]_0^{\sqrt{3}} = \sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

The inverse tangent function: although not one-to-one on its full domain, the tangent function is one-to-one on the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and on that interval it takes on as a value every real number. Thus, if x is real, there is one and only one number in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ at which the tangent function takes on the value x . This number is called the inverse tangent of x and is written $\tan^{-1} x$ or $\arctan x$. The inverse tangent function

$$y = \tan^{-1} x, \quad x \in \mathbf{R}$$

is the inverse of the function

$$y = \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of these two functions are the reflection of each other about the line $y = x$. While the tangent has vertical asymptotes, the inverse tangent has horizontal asymptotes. Both functions are odd functions. Because they are inverses, for every real number x , $\tan(\tan^{-1} x) = x$ and for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\tan^{-1}(\tan x) = x$. It is hard to make a mistake with the first relation since it applies for all real numbers. The second relation requires the usual care: $\tan^{-1}(\tan \frac{7\pi}{5}) = \tan^{-1}(\tan(\frac{2\pi}{5} + \pi)) = \tan^{-1}(\tan \frac{2\pi}{5}) = \frac{2\pi}{5}$

We find the derivative as we did for the inverse sine:

$$y = \tan^{-1} x \implies \tan y = x \implies \sec^2 y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

1. calculate $\frac{d}{dx}[\tan^{-1}(ax^2 + bx + c)]$

By chain rule we have $\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2} \frac{du}{dx}$ so $\frac{d}{dx}[\tan^{-1}(ax^2 + bx + c)] = \frac{1}{1+(ax^2 + bx + c)^2} \frac{d}{dx}(ax^2 + bx + c) = \frac{2ax + b}{1+(ax^2 + bx + c)^2}$

2. Show that $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

We change variables so that a^2 is replaced by 1 and we can use our formula. We set $au = x \implies adu = dx \implies \int \frac{dx}{a^2 + x^2} = \int \frac{adu}{a^2 + a^2u^2} = \frac{1}{a} \int \frac{du}{1 + u^2} = \frac{1}{a} \tan^{-1} u + c = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

3. evaluate $\int_0^2 \frac{dx}{4 + x^2}$

$\int \frac{dx}{4 + x^2} = \int \frac{dx}{2^2 + x^2} = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c \implies \int_0^2 \frac{dx}{4 + x^2} = \left[\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) \right]_0^2 = \frac{1}{2} \tan^{-1} 1 = \frac{\pi}{8}$

Derivatives of Inverse Trig Functions:

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

2. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

4. $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

6. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

More Examples

1. Find the exact value of the following:

(a) $\csc^{-1} 2$

$$\csc \frac{\pi}{6} = 2 \text{ and } \frac{\pi}{6} \text{ is in the appropriate interval so } \csc^{-1} 2 = \frac{\pi}{6}$$

(b) $\tan^{-1}(-1)$

$$\tan \frac{\pi}{4} = 1 \implies \tan -\frac{\pi}{4} = -1 \implies \tan^{-1}(-1) = -\frac{\pi}{4} \text{ (remember to check the interval.)}$$

(c) $\sec^{-1} \sqrt{2}$

$$\sec \frac{\pi}{4} = \sqrt{2} \implies \sec^{-1} \sqrt{2} = \frac{\pi}{4} \text{ (remember to check the interval)}$$

(d) $\sin^{-1} 1$

$$\sin \frac{\pi}{2} = 1 \implies \sin^{-1} 1 = \frac{\pi}{2} \text{ (remember to check the interval)}$$

(e) $\tan(\cos^{-1} 1/2)$

$$\cos^{-1} 1/2 = \frac{\pi}{3} \implies \tan \frac{\pi}{3} = \sqrt{3}$$

(f) $\sin^{-1}(\sin 1)$

$$= 1 \text{ because } -\pi/2 \leq 1 \leq \pi/2$$

2. Differentiate the following:

(a) $y = \sin^{-1}(x^2)$

$$\implies y' = \frac{1}{\sqrt{1 - (x^2)^2}} \frac{d}{dx} x^2 = \frac{2x}{\sqrt{1 - x^4}}$$

(b) $y = (1 + x^2) \tan^{-1} x$

$$y' = 2x \tan^{-1} x + (1 + x^2) \frac{1}{1 + x^2} = 1 + 2x \tan^{-1} x$$

(c) $y = \tan^{-1}(\cos x)$

$$y' = \frac{1}{(\cos x)^2} (-\sin x) = -\frac{\sin x}{1 + \cos^2 x}$$

3. Integrate the following:

(a) $\int_1^{\sqrt{3}} \frac{6}{1 + x^2} dx$

$$= [6 \tan^{-1} x]_1^{\sqrt{3}} = 6(\tan^{-1} \sqrt{3} - \tan^{-1} 1) = 6 \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\pi}{2}$$

$$(b) \int \frac{x}{1-x^4} dx$$

$$\text{Let } u = x^2 \implies du = 2x dx \implies \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{2} \sin^{-1} u + c = \frac{1}{2} \sin^{-1}(x^2) + c$$

$$(c) \int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$$

$$\text{Let } u = \sin^{-1} x \implies du = \frac{dx}{\sqrt{1-x^2}} \implies \int u du = \frac{1}{2} u^2 = \left[\frac{1}{2} (\sin^{-1} x)^2 \right]_0^{1/2} = \frac{\pi^2}{72}$$

$$(d) \int \frac{1}{x\sqrt{x^2-4}} dx$$

$$\text{Let } u = \frac{x}{2} \implies du = \frac{dx}{2} \implies \int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{2x\sqrt{(x/2)^2-1}} = \int \frac{2du}{4u\sqrt{u^2-1}} = \frac{1}{2} \int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2} \sec^{-1} u + c = \frac{1}{2} \sec^{-1} \left(\frac{x}{2} \right) + c$$

$$(e) \int \frac{e^x}{e^{2x}+1} dx$$

$$\text{Let } u = e^x \implies du = e^x dx \implies \int \frac{du}{u^2+1} = \tan^{-1} u + c = \tan^{-1}(e^x) + c$$