

Lecture notes 8.1: Integration By Parts

Because of FTC, we can integrate a function if we know an antiderivative. These are the formulas we know so far:

$$\begin{array}{ll}
 \int x^n dx = \frac{x^{n+1}}{n+1} + c (n \neq -1) & \int \frac{1}{x} dx = \ln|x| + c \\
 \int e^x dx = e^x + c & \int a^x dx = \frac{a^x}{\ln a} + c \\
 \int \sin x dx = -\cos x + c & \int \cos x dx = \sin x + c \\
 \int \sec^2 x dx = \tan x + c & \int \csc^2 x dx = -\cot x + c \\
 \int \sec x \tan x dx = \sec x + c & \int \csc x \cot x dx = -\csc x + c \\
 \int \tan x dx = \ln|\sec x| + c & \int \cot x dx = \ln|\sin x| + c \\
 \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c & \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c
 \end{array}$$

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

$$\begin{aligned}
 \frac{d}{dx}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x) \\
 \implies \int [f(x)g'(x) + g(x)f'(x)] dx &= f(x)g(x) \\
 \implies \int f(x)g'(x) dx + \int g(x)f'(x) dx &= f(x)g(x) \\
 \implies \int f(x)g'(x) dx &= f(x)g(x) - \int g(x)f'(x) dx \\
 \text{Let } f(x) = u \text{ and } v = g(x) \implies du = f'(x), dv = g'(x) & \\
 \implies \int u dv = uv - \int v du &
 \end{aligned}$$

This is our formula for integration by parts.

Find $\int x \sin x dx$

$$\begin{aligned}
 \text{Let } u = x, v' = \sin x \implies du = dx, v = -\cos x & \\
 \implies \int x \sin x dx = u dv - \int v du = x(-\cos x) - \int (-\cos x) dx &= -x \cos x + \\
 \int \cos x dx = -x \cos x + \sin x + c &
 \end{aligned}$$

when using integration by parts, it is a good idea to check your answers by differentiating. When using integration by parts, the goal is to obtain a simpler integral than the one we started with. In general, when deciding on a

choice for u and dv , try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)dx$ can be integrated to give v . Examples:

$$1. \int x e^{2x} dx$$

$$u = x, dv = e^{2x} dx \implies du = dx, v = \frac{1}{2} e^{2x}$$

$$\begin{aligned} \implies \int x e^{2x} dx &= \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx \\ &= \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + c \end{aligned}$$

$$2. \int x \sin 4x dx$$

$$u = x, dv = \sin 4x dx \implies du = dx, v = -\frac{1}{4} \cos 4x$$

$$\begin{aligned} \int x \sin 4x dx &= -\frac{x}{4} \cos 4x - \int -\frac{1}{4} \cos 4x dx \\ &= -\frac{x}{4} \cos 4x + \int \frac{1}{4} \cos 4x dx \\ &= -\frac{x}{4} \cos 4x + \frac{1}{16} \sin 4x + c \end{aligned}$$

$$3. \int_1^2 \frac{\ln x}{x^2} dx$$

$$u = \ln x, dv = x^{-2} dx \implies du = x^{-1} dx, v = -x^{-1}$$

$$\begin{aligned} \implies & -\frac{1}{x} \ln x - \int -\frac{1}{x} \frac{1}{x} dx \\ &= -\frac{1}{x} \ln x + \int x^{-2} dx \\ &= \left[-\frac{1}{x} \ln x - x^{-1} \right]_1^2 \\ &= -\frac{1}{2} \ln 2 - \frac{1}{2} - (-\ln 1 - 1) \\ &= -\frac{1}{2} \ln 2 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2} \ln 2 \end{aligned}$$

$$4. \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} x^3 \cos(x^2) dx$$

$$\text{let } m = x^2 \implies dm = 2x dx \implies \frac{1}{2} dm = x dx \implies \int \frac{1}{2} m \cos m dm$$

$$u = \frac{1}{2} m, dv = \cos m dm \implies du = \frac{1}{2} dm, v = \sin m$$

$$\begin{aligned}
&\implies \frac{m}{2} \sin m - \int \frac{1}{2} \sin m dm \\
&= \frac{m}{2} \sin m + \frac{1}{2} \cos m \\
&\implies \left[\frac{x^2}{2} \sin(x^2) + \frac{1}{2} \cos(x^2) \right]_{\sqrt{\pi/2}}^{\sqrt{\pi}} \\
&= \frac{\pi}{2} \sin \pi + \frac{1}{2} \cos \pi - \left(\frac{\pi}{4} \sin(\pi/2) + \frac{1}{2} \cos(\pi/2) \right) \\
&= 0 + 1/2(-1) - (\pi/4 + 0) = -\frac{1}{2} - \frac{\pi}{4}
\end{aligned}$$

5. Prove $\int x^n e^x dx = x^n - n \int x^{n-1} e^x dx$

$$\begin{aligned}
u = x^n, dv = e^x dx &\implies du = nx^{n-1} dx, v = e^x \implies \int x^n e^x dx = x^n - \\
&n \int x^{n-1} e^x dx
\end{aligned}$$