

## Lecture notes 12.4: The Comparison Tests

The general idea is to compare a given series with a series that is known to be convergent or divergent.

The comparison test: Suppose that  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are series with positive terms,

1. if  $\sum_{n=1}^{\infty} b_n$  is convergent and  $a_n \leq b_n$  for all  $n$ , then  $\sum_{n=1}^{\infty} a_n$  is also convergent
2. if  $\sum_{n=1}^{\infty} b_n$  is divergent and  $a_n \geq b_n$  for all  $n$ , then  $\sum_{n=1}^{\infty} a_n$  is also divergent

To use this test, we need a known series for comparison. Usually we use either a p-series or geometric series.

Determine whether the series  $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$  converges or diverges.

For large  $n$  the dominant term in the denominator is  $2n^2$  so we compare the given series with the series  $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ .

$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$  and we know  $\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent because it's a constant times a p-series with  $p = 2 > 1$ . Hence our series is convergent by the comparison test.

Test the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  for convergence or divergence.

$\frac{\ln n}{n} > \frac{1}{n}$  for  $n \geq 3$  and we know  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent (p-series with  $p = 1$ ).

Hence our series is divergent by the comparison test.

Note: if the terms of a series are larger than those of a convergent series, or smaller than those of a divergent series, it tells us nothing. The inequalities have to go in the right direction.

The limit comparison test: Suppose that  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are series with positive terms. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$  with  $0 < c < \infty$ , then either both series converge or both series diverge.

Test the series  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  for convergence or divergence.

Let  $a_n = \frac{1}{2^n - 1}$  and  $b_n = \frac{1}{2^n}$  and use the limit comparison test.

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/2^n} = 1 > 0$  Since this limit exists and

$\sum_{n=1}^{\infty} \frac{1}{2^n}$  is a convergent geometric series, the given series converges by the limit comparison test.

Determine whether the series  $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$  converges or diverges.

the dominant part of the numerator is  $2n^2$  and the dominant part of the denominator is  $\sqrt{n^5} = n^{5/2}$ . This suggests taking

$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$  and  $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(2n^2 + 3n)n^{1/2}}{(\sqrt{5 + n^5})2} = \lim_{n \rightarrow \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}} = \lim_{n \rightarrow \infty} \frac{2 + (3/n)}{2\sqrt{(5/n^5) + 1}} =$

1. Since  $\sum_{n=1}^{\infty} b_n = 2 \sum_{n=1}^{\infty} 1/n^{1/2}$  is divergent ( $p$ -series with  $p = 1/2 < 1$ ), the given series diverges by the limit comparison test.

Note: in testing many series we find a suitable comparison series by keeping only the highest powers in the numerator and denominator.