Lecture notes 9.1: Arc Length

The arc length formula: If f' is continuous on $[a,b]$, then the length of the curve $y=f(x), a \le x \le b$ is

$$L=\int_a^b \sqrt{1+[f'(x)]^2}dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

If a curve has the equation $x=g(y), c \le y \le d$ and g' is continuous, then

$$L=\int_c^d \sqrt{1+[g'(y)]^2}dy=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$

Examples

1. Find the length of the arc of the parabola $y^2=x$ from $(0,0)$ to $(1,1)$

   Solution: $x=y^2 \Longrightarrow \frac{dx}{dy}=2y \Longrightarrow L=\int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}dy=\int_0^1 \sqrt{1+4y^2}dy$

   let $y=\frac{1}{2}\tan\theta \Longrightarrow dy=\frac{1}{2}\sec^2\theta d\theta$

   $\Longrightarrow \sqrt{1+4y^2}=\sqrt{1+\tan^2\theta}=\sec\theta$

   $y=0 \Longrightarrow \tan\theta=0 \Longrightarrow \theta=0, y=1 \Longrightarrow \tan\theta=2 \Longrightarrow \theta=\tan^{-1}2$

   $\int_0^{\tan^{-1}2} \sec\theta\sec^2\theta d\theta=\frac{1}{2}\int_0^{\tan^{-1}2}\sec^3\theta d\theta$

   $=\frac{1}{4}[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|]_0^{\tan^{-1}2}$

2. Find the length of the arc of $y^2=x^3$ between $(1,1)$ and $(4,8)$

   Solution: $y=x^{3/2} \Longrightarrow \frac{dy}{dx}=\frac{3}{2}x^{1/2} \Longrightarrow L=\int_1^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}dx=\int_1^4 \sqrt{1+\frac{9}{4}x}dx$

   $u=1+\frac{9x}{4} \Longrightarrow du=\frac{9}{4}dx$

   $x=1 \Longrightarrow u=\frac{13}{4}, x=4 \Longrightarrow u=10$

   $L=\int_{13/4}^{10}\frac{4}{9}u^{1/2}du=\frac{4}{9}\frac{2}{3}[u^{3/2}]_{13/4}^{10}=\frac{8}{27}\left[10^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right]$

Arc Length function: $s(x) = \int_a^x \sqrt{1 + [f'(x)]^2} dt$

For a smooth curve $y = f(x), a \leq x \leq b$, $s(x)$ is the distance along the curve from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.

Example: Find the arc length function for the curve $y = x^2 - (\ln x)/8$ taking $P_0(1, 1)$ as the starting point.

Solution: $f(x) = x^2 - (\ln x)/8 \Longrightarrow f'(x) = 2x - \dfrac{1}{8x}$

$\Longrightarrow 1 + [f'(x)]^2 = 1 + \left(2x - \dfrac{1}{8x}\right)^2 = 1 + 4x^2 - \dfrac{1}{2} + \dfrac{1}{64x^2} = 4x^2 + \dfrac{1}{2} + \dfrac{1}{64x^2} = \left(2x + \dfrac{1}{8x}\right)^2$

$\sqrt{1 + [f'(x)]^2} = 2x + \dfrac{1}{8x}$

$s(x) = \int_1^x \left(2t - \dfrac{1}{8t}\right) dt = \left[t^2 + \dfrac{1}{8} \ln t\right]_1^x = x^2 + \dfrac{1}{8} \ln x - 1$

So the arc length along the curve from $(1, 1)$ to $(3, f(3))$ is $s(3) = 3^2 + \dfrac{1}{8} \ln 3 - 1$

Example: A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ feet is given by $y = 150 - \dfrac{1}{40}(x - 50)^2$. Find the distance traveled by the kite.

Solution: $y = 150 - \dfrac{1}{40}(x - 50)^2 \Longrightarrow y' = -\dfrac{1}{20}(x - 50) \Longrightarrow 1 + (y')^2 = 1 + \dfrac{1}{20^2}(x - 50)^2$

so the distance traveled by the kite is

$L = \int_0^{80} \sqrt{1 + \dfrac{1}{20^2}(x - 50)^2} dx$

$u = \dfrac{1}{20}(x - 50), du = \dfrac{1}{20} dx$

$\Longrightarrow \int_{-5/2}^{3/2} \sqrt{1 + u^2}\, (20du)$