Lecture notes: 6.4: Work

We begin with a constant force F that acts along some line that we call the x-axis. By convention F is positive if it acts in the direction of increasing x and negative if it acts in the direction of decreasing x. Suppose that an object moves along the x-axis from $x = a$ to $x = b$ subject to this constant force F. The **work** done by F during the displacement is by definition the force times displacement:

$$W = F(b - a)$$

We see that if F acts in the direction of the motion, then $W > 0$, but if F acts in the against the motion, $W < 0$. Take, for example, a book and drop it to the floor. The motion towards the floor is the same as the direction of the force of gravity and so the work is positive. But if you lift the same book from the floor, the motion is opposite the direction of the force of gravity and so the work done by gravity is negative. However, the force of the hand lifting the book is in the same direction as the motion so the work done by the hand is positive.

So what if the force F is not constant? i.e. if F varies continuously as function of x? Then we define the work done by F as the **average value** of F times $b - a$:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x = \int_a^b F(x)dx$$

Hooke's Law: You can sense a variable force in the action of a steel spring. Stretch a spring within its elastic limit and you feel a pull in the opposite direction. The greater you stretch it, the harder the pull of the spring. Conversely, compress a spring within its elastic limit and you feel a push against you. According to Hooke's Law, the force exerted by the spring can be written $F(x) = -kx$ where k is a positive number called the spring constant and x is the displacement from the equilibrium configuration. The minus sign indicates the spring force always acts in the opposite direction which the spring has been deformed (i.e. the force acts to restore the spring to equilibrium). Note: Hooke's Law is only an approximation but it's good for small displacements.

Examples:

1. When a particle is located at a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Solution: $W = \int_1^3 F(x)dx = \int_1^3 (x^2+2x)dx = \left[\frac{x^3}{3}+x^2\right]_1^3 = \frac{50}{3}$ So the work done is $16\frac{2}{3}$ foot-pounds.

2. A spring of natural length L compressed to length $\frac{7}{8}L$ exerts a force F_0. Find the work done by the spring in restoring itself to natural length.

   Solution: Place the spring on the x-axis so the equilibrium point falls at the origin and think of compression as a move to the left. Compressed $\frac{1}{8}L$ units to the left, the spring exerts a force F_0. Thus by Hooke's Law:

$$F_0 = F\left(-\frac{1}{8}L\right) = -k\left(-\frac{1}{8}L\right) = \frac{1}{8}kL$$

   This tells us that $k = 8F_0/L$. Therefore, the force law for this spring reads:

$$F(x) = -\left(\frac{8F_0}{L}\right)x$$

   To find the work done by this spring in restoring itself to equilibrium, we integrate $F(x)$ from $x = -\frac{1}{8}L$ to $x = 0$:

$$W = \int_{-L/8}^{0} F(x)dx = \int_{-L/8}^{0} -\left(\frac{8F_0}{L}\right)xdx = -\frac{8F_0}{L}\left[\frac{x^2}{2}\right]_{-L/8}^{0} = \frac{LF_0}{16}$$

3. For the same spring, what work must we do to stretch the spring to length $\frac{11}{10}L$?

   Solution: To stretch the spring we must counteract the force of the spring. The force exerted by the spring when stretched x units is

$$F(x) = -\left(\frac{8F_0}{L}\right)x$$

   To counter this force we must apply the opposite force

$$-F(x) = \left(\frac{8F_0}{L}\right)x$$

The work we must do to stretch the spring to length $\frac{11}{10}L$ can be found by integrating $-F(x)$ from $x = 0$ to $x = \frac{1}{10}L$:

$$W = \int_0^{L/10} -F(x)dx = \int_0^{L/10} \left(\frac{8F_0}{L}\right) xdx = \frac{8F_0}{L}\left[\frac{x^2}{2}\right]_0^{L/10} = \frac{LF_0}{25}$$

Note: if force is measured in pounds and distance in feet, the units of work are foot-pounds. So if a force of 500 pounds pushes a car for 60 feet, the work done is 30,000 foot-pounds. If force is given in Newtons and distance measured in meters, then work is given in Newton-meters, also called Joules. If force is given in dynes and distance in centimeters, then work is given in dyne-centimeters, called ergs.

4. Stretched $\frac{1}{3}$ foot beyond its natural length, a certain spring exerts a restoring force of 10 pounds. What work must we do to stretch the spring another $\frac{1}{3}$ foot?

   Solution: Place the spring on the x-axis so that the equilibrium point falls at the origin and think of stretching as a move to the right. We assume Hooke's Law: $F(x) = -kx$. When the spring is stretched $\frac{1}{3}$ foot, it exerts a force of -10 pounds (10 pounds to the left). So $-10 = -k\left(\frac{1}{3}\right)$ and $k = 30$ (i.e. 30 pounds per foot). To find the work we must do to stretch the spring another $\frac{1}{3}$ foot, we integrate the opposite force $-F(x) = 30x$ from $x = \frac{1}{3}$ to $x = \frac{2}{3}$:

$$W = \int_{1/3}^{2/3} 30xdx = 30\left[\frac{1}{2}x^2\right]_{1/3}^{2/3} = 5$$

   where our units are foot-pounds.

Pumping out a tank: To lift an object we must counteract the force of gravity. Consequently, the work done in lifting an object is given by the formula work=(weight of the object)x(distance lifted). If we lift a leaking sand bag or pump out a water tank from above, the calculation of work is more complicated. In the first instance the weight varies during the motion (there is less sand in the bag as we keep lifting). In the second instance the distance varies (water at the top of the tank does not have to be pumped as far as water at the bottom).

Suppose we have a storage tank filled to within a feet of the top with some liquid, and we assume the liquid is homogeneous and weighs σ pounds per cubic foot. Suppose now that this storage tank is pumped out from above until the level of the liquid drops to b feet below the top of the tank. How much work has been done?

Solution: We can answer this question by the methods of integral calculus. For each $x \in [a, b]$, we let

$A(x)$ =cross-sectional area x feet below the top of the tank,

$s(x)$ =distance that the x-level must be lifted.

We let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$ and focus our attention on the ith subinterval $[x_{i-1}, x_i]$. Taking x_i^* as an arbitrary point in the ith subinterval, we have

$A(x_i^*)\triangle x_i$ =approximate volume of the ith layer of liquid,

$\sigma A(x_i^*)\triangle x_i$ =approximate weight of this volume,

$s(x_i^*)$ =approximate distance this weight is to be moved,

and therefore

$\sigma s(x_i^*)A(x_i^*)\triangle x_i$ =approximate work (weight x distance) required to pump this layer of liquid to the top of the tank.

The work required to pump out all the liquid can be approximated by adding up all these last terms:

$$W = \sigma s(x_1^*)A(x_1^*)\triangle x_1 + \sigma s(x_2^*)A(x_2^*)\triangle x_2 + \ldots + \sigma s(x_n^*)A(x_n^*)\triangle x_n$$

Which we see are Reimann Sums that converge to give us:

$$W = \int_a^b \sigma s(x)A(x)dx$$

Example: A hemispherical water tank of radius 10 feet is being pumped out. Find the work done in lowering the water level from 2 feet below the top of the tank to 4 feet below the top of the tank given that tthe pump is placed (a) at the top of the tank and (b) 3 feet above the top of the tank.

Solution: As the weight of water, take 62.5 pounds per cubic foot. It is not hard to see that the cross section x feet from the top of the tank is a disc of radius $\sqrt{100 - x^2}$. Its area is therefore

$$A(x) = \pi(100 - x^2)$$

For part (a) we have $s(x) = x$, so that

$$W = \int_2^4 62.5\pi x(100 - x^2)dx = 33750\pi$$

For part (b) we have $s(x) = x + 3$, so that

$$W = \int_2^4 62.5\pi(x + 3)(100 - x^2)dx = 67750\pi$$

Homework: all assignments from 6.4