Name Solutions

Z-Number ____________________

MATH 434 / 534 (EXAM 2)

Professor B. Datta
FALL 2011

Note: Each problem carries 20 points.

1. (a) Show that the matrix

$$A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

is both **diagonally dominant** and **symmetric positive definite**.

In order to show that A is diagonally dominant,

$$4 > |-1| + |-1| = 2$$
$$4 > |-1| + |0| = 1$$
$$4 > |0| + |-1| = 1.$$

$\Rightarrow A$ diagonally dominant.
To show that A is symmetric positive definite, check the determinant of each leading submatrix is positive. The leading principle minors are 4, 15, and 56.

(b) Find the **Cholesky factorization** of A.

$$h_{11} = \sqrt{a_{11}} = 2.$$
$$h_{21} = \frac{a_{21}}{h_{11}} = -0.5.$$
$$h_{22} = \sqrt{a_{22} - h_{21}^2} = 1.9365.$$
$$h_{31} = \frac{a_{31}}{h_{11}} = -0.5.$$
$$h_{32} = \frac{a_{32} - h_{21}h_{31}}{h_{22}} = -0.1291.$$
$$h_{33} = \sqrt{a_{33} - h_{31}^2 - h_{32}^2} = 1.9322.$$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ -0.5 & 1.9365 & 0 \\ -0.5 & -0.1291 & 1.9322 \end{pmatrix}$$

(c) Using **the Cholesky factorization**, solve the system:

$$Ax = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 1:

$$\begin{aligned} A &= HH^T \\ Ax &= HH^T x = b \end{aligned}$$

Step 2:

$$Hy = b$$

Step 3:

$$H^T x = y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(d) Compute $\det(A)$ **from the Cholesky factorization**. [No credit will be given if you do otherwise.]

Since a matrix A is decomposed as $A = HH^T$ and H is a triangular matrix

$$\begin{aligned} \det(A) &= \det(H)\det(H^T) \\ &= \det(H)\det(H) = 56 \end{aligned}$$

2. (a) Describe an algorithm to compute a Householder matrix H such that

$$H \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$$

$$u = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}$$

$$H = I - 2\frac{uu^T}{u^T u} = \begin{pmatrix} 0 & -0.7071 & -0.7071 \\ -0.7071 & 0.5 & -0.5 \\ -0.7071 & -0.5 & 0.5 \end{pmatrix}$$

Then we have

$$H \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.442 \\ 0 \\ 0 \end{pmatrix}.$$

(b) Using Householder's method, find the QR factorization of

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Step 1.

$$H_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \star \\ 0 \\ 0 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} 0 & -0.7071 & -0.7071 \\ -0.7071 & 0.5 & -0.5 \\ -0.7071 & -0.5 & 0.5 \end{pmatrix}$$

$$H_1 A = A^{(1)} = \begin{pmatrix} -1.4142 & -3.5355 \\ 0 & -1.2071 \\ 0 & -0.2071 \end{pmatrix}$$

Step 2.

$$\widehat{H_2}\begin{pmatrix}-1.2071\\-0.2071\end{pmatrix}=\begin{pmatrix}\star\\0\end{pmatrix}$$

$$\widehat{u_2}=\begin{pmatrix}-1.2071\\-0.2071\end{pmatrix}-\left\|\begin{pmatrix}=-1.2071\\-0.2071\end{pmatrix}\right\|e_1$$

$$\widehat{H_2}=\begin{pmatrix}-0.9856 & -0.1691\\-0.1691 & 0.9856\end{pmatrix}$$

Step 3.

$$H_2=\begin{pmatrix}1 & 0 & 0\\0 & \widehat{H_2}(1,1) & \widehat{H_2}(1,2)\\0 & \widehat{H_2}(2,1) & \widehat{H_2}(2,2)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\0 & -0.9856 & -0.1691\\0 & -0.1691 & 0.9856\end{pmatrix}$$

$$Q=H_1H_2=\begin{pmatrix}0 & 0.8165 & -0.5774\\-0.7071 & -0.4082 & -0.5774\\-0.7071 & 0.4082 & 0.5774\end{pmatrix}$$

Step 4.

$$R=H_2H_1A=\begin{pmatrix}-1.4142 & -3.5355\\0 & 1.2297\\0 & 0\end{pmatrix}$$

(c) Find the least-square solution to: $Ax=\begin{pmatrix}1\\3\\4\end{pmatrix}$ **using the QR factorization**.

Step 1.

$$QRx=b \text{ or } Rx=Q^Tb=c$$
$$Rx=\begin{pmatrix}-4.9497\\1.2247\\0\end{pmatrix}$$

Step 2.

$$Rx=c \text{ for } x,$$
$$\text{we have } x=(1,1)^T$$

3. (a) State a mathematical result that shows that solving least-squares problem by QR factorization is a stable process.

Computed solution $\hat{x}$ is such that it minimizes $\|(A+E)\hat{x} - (b+\delta b)\|_2$, where E and δb are small. Specially,

$$\begin{aligned} \|E\|_F &\le c\mu n\|A\|_F + O(\mu^2) \\ \|\delta b\|_2 &\le c\mu\|b\|_2 + O(\mu^2) \end{aligned}$$

where $c \approx 6m - 3n + 41$ and μ is the machine precision.

(b) State a **mathematical result** that shows that Householder's method for QR factorization is stable.

If $\hat{R}$ denotes the computed R, then $\exists$ an orthogonal $\hat{Q}$ such that $A + E = \hat{Q}\hat{R}$. The error matrix E satisfies $\|E\|_F \le \phi(n)\mu\|A\|_F$ where $\phi(n)$ is a slowly growing function of n and μ is the machine precision.

4. Given

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) Find the singular values and singular vectors of A.

SVD of A :

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \quad V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Left Singular Vectors : $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Right Singular Vectors : $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) Find $\|A\|_2, \|A\|_F, Cond_2(A)$, **using the singular values of** A.

$$\begin{aligned} \|A\|_2 &= \sigma_{\max} = 2. \\ \|A\|_F &= \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{5}. \\ cond_2(A) &= \frac{\sigma_{\max}}{\sigma_{\min}} = \frac{2}{1} = 2. \end{aligned}$$

(c) Find the least-squares solution to $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, using the singular value decomposition of A.

Step 1. Deleting the last column of U and last row of Σ, we obtained a truncated SVD of A as follows :

$$\bar{U}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \Sigma_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \quad V_1^T = V_1^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Step 2.

$$b' = \bar{U}_1^T b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 3.

$$\Sigma_1 y = b'$$
$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$y_1 = \frac{1}{2}, \quad y_2 = 1.$$

Step 4.

$$x = V_1 y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

(d) Construct an example of an 3×2 matrix which is of full rank (theoretically) but close to a rank-deficient matrix.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-4} \\ 0 & 0 \end{pmatrix}$$

5. Prove the following results:

(a) $\|A^TA\|_2 = \|A\|_2^2$

$$\begin{aligned} A &= \bar{U}\Sigma V^T \to \|A\|_2 = \sigma_{\max} = \sigma_1 \ldots \mathbf{1} \\ A^T &= V\Sigma\bar{U}^T \\ \text{So, } A^TA &= \bar{U}\Sigma^T V V^T \Sigma \bar{U} = \bar{U}\Sigma\Sigma^T\bar{U}^T. \\ \text{Since } \Sigma\Sigma_2^T &= \text{ diag } (\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2), \\ \|A^TA\|_2 &= \sigma_{\max}^2 = \sigma_1^2 \ldots \mathbf{2} \to \\ \mathbf{1} \ \& \ \mathbf{2} &\to \|A\|_2^2 = \|A^TA\|_2. \end{aligned}$$

(b) $||A||_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}$, when $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of A.

$$\begin{aligned} \text{Since } \|A\|_F^2 &= \text{ trace } (A^TA) \text{ and } A = \bar{U}\Sigma V^T; \\ \|A\|_F^2 &= \text{ trace } (A^TA) = \text{ trace } (V\Sigma^T F\bar{U}\Sigma V^T) \\ &= \text{ trace } (V\Sigma^T\Sigma V^T) \end{aligned}$$

Since two similar matrices have the same eigenvalues and the trace of a matrix is the sum of eigenvalues,

$$\begin{aligned} \|A\|_F^2 &= \text{ trace } (V\Sigma^T\Sigma V^T) = \text{ trace } (\Sigma^T\Sigma) \\ &= \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2. \end{aligned}$$

(c) rank $(A^TA) =$ rank (A).

$$A = \bar{U}\Sigma V^T \qquad A^T = V\Sigma\bar{U}^T$$

$$m \geq n. \qquad A = \bar{U}\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \end{pmatrix}$$

$$A^TA = V\sigma^T\Sigma V^T$$

$$\Sigma^T\Sigma = \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{pmatrix}_{n\times n}$$

The # of nonzero singular values of A is equal to the # of nonzero singular values of $A^T A \to \text{rank}\ (A) = \text{rank}\ (A^T A)$.