# *Lecture Notes on Numerical Differential Equations: IVP*

**Professor Biswa Nath Datta**

**Department of Mathematical Sciences**
**Northern Illinois University**
**DeKalb, IL. 60115 USA**

**E−mail: dattab@math.niu.edu**

**URL: www.math.niu.edu/~dattab**

# 1 Initial Value Problem for Ordinary Differential Equations

We consider the problem of numerically solving a system of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha \text{ (given)} .$$

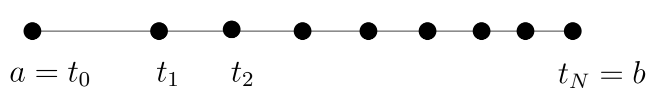
Such a problem is called the **Initial Value Problem** or in short **IVP**, because the initial value of the solution $y(a) = \alpha$ is given.

Since there are infinitely many values between $a$ and $b$, we will only be concerned here to find approximations of the solution $y(t)$ at several specified values of $t$ in $[a, b]$, rather than finding $y(t)$ at every value between $a$ and $b$.

Denote

- $y_i$ = An approximation of $y(t_i)$ at $t = t_i$.
- Divide $[a, b]$ into $N$ equal subintervals of length $h$:

$$t_0 = a < t_1 < t_2 < \cdots t_N = b.$$




- $h = \dfrac{b-a}{N}$ **(step size)**

**Notation:**

$$y(t_i) \equiv \text{ Exact value at } t = t_i.$$
$$y_i \equiv \text{ Approximate value of } y(t_i).$$

**The Initial Value Problem**
**Given**

(1) $y' = f(y, t), \ a \leq t \leq b$

(2) The initial value $y(t_0) = y(a) = \alpha$

(3) The step-size $h$.

**Find** $y_i$ (approximate value of $y(t_i)$), $i = 1, \cdots, N$, where $N = \dfrac{b-a}{h}$.

We will briefly describe here the following well-known numerical methods for solving the IVP:

- The **Euler Method**
- The **Taylor Method** of higher order
- The **Runge-Kutta Method**
- The **Adams-Moulton Method**
- The **Milne Method**

etc.

We will also discuss the error behavior and convergence of these methods.

However, before doing so, we state a result **without proof**, in the following section on the **existence** and **uniqueness** of the solution for the IVP. The proof can be found in most books on ordinary differential equations.

**Existence and Uniqueness of the Solution for the IVP**

**Theorem: (Existence and Uniqueness Theorem for the IVP).**

The initial value problem:

$$\begin{cases} y' = f(t, y) \\ y(a) = \alpha \end{cases}$$

has a unique solution $y(t)$ for $a \le t \le b$, if $f(t, y)$ is continuous on the domain, given by $R = \{a \le t \le b, \quad \infty < y < \infty\}$ and satisfies the following inequality:

$$|f(t, y) - f(t, y^*)| \le L|y - y^*|,$$

Whenever $(t, y)$ and $(t, y^*) \in R$.

$\blacksquare$

**Definition**. The condition $|f(t, y) - f(t, y^*)| \le L|y - y^*|$ is called the **Lipschitz Condition**. The number $L$ is called a **Lipschitz Constant**.

**Definition.**

A set $S$ is said to be convex if whenever $(t_1, y_1)$ and $(t_2, y_2)$ belong to $S$, the point $((1 - \lambda)t_1 + \lambda t_2, \ (1 - \lambda)y_1 + \lambda y_2)$ also belongs to $S$ for each $\lambda$ when $0 \le \lambda \le 1$.

**Simplification of the Lipschitz Condition for the Convex Domain**

If the domain happens to be a **convex set**, then the condition of the above Theorem reduces to

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \le L \text{ for all } (t, y) \in R.$$

## Liptischitz Condition and Well-Posednes

**Definition.**

An IVP is said to be **well-posed** if a small perhubation in the data of the problem leads to only a small change in the solution.

Since numerical computation may very well introduce some perhubations to the problem, it is important that the problem that is to be solved is well-posed.

Fortunately, the Lipschitz condition is a sufficient condition for the IVP problem to be well-posed.

**Theorem** (**Well-Posedness of the IVP problem**).
If $f(t, y)$ Satisfies the Lipschitz Condition, then the IVP is well-posed.

## 2 The Euler Method

One of the simplest methods for solving the IVP is the classical Euler method.

The method is derived from the Taylor Series expansion of the function $y(t)$.

The function $y(t)$ has the following Taylor series expansion of order $n$ at $t = t_{i+1}$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2!} y''(t_i) + \cdots + \frac{(t_{i+1} - t_i)^n}{n!} y^{(n)}(t_i) +$$

$$\frac{(t_{i+1} - t_i)^{n+1}}{(n+1)!} y^{n+1}(\xi_i),$$ where $\xi_i$ is in $(t_i, t_{i+1})$.

Substitute $h = t_{i+1} - t_i$. Then

**Taylor Series Expansion of $y(t)$ of order $n$ at $t = t_{i+1}$:**

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i).$$

For $n = 1$, this formula reduces to

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi).$$

The term $= \dfrac{h^2}{2!} y^{(2)}(\xi_i)$ is call the **remainder term**.

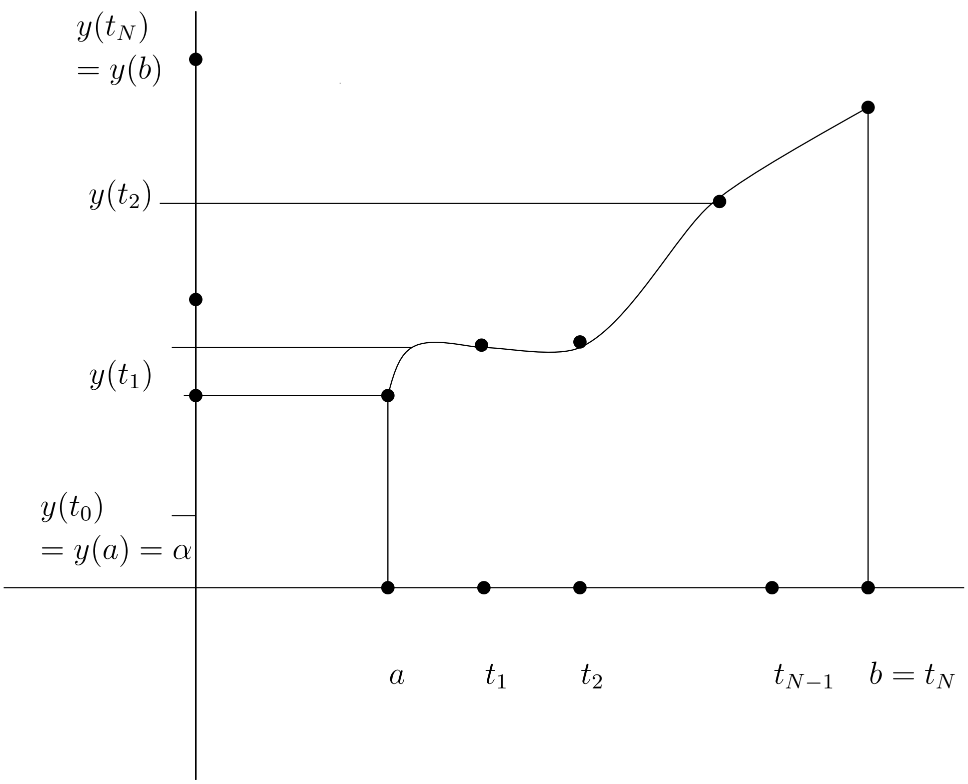
Neglecting the remainder term, we have

**Euler's Method**

$$\begin{aligned} y_{i+1} &= y_i + h y'(t_i) \\ &= y_i + h f(t_i, y_i), \quad i = 0, 1, 2, \cdots, N-1 \end{aligned}$$

This formula is known as the **Euler method** and now can be used to approximate $y(t_{i+1})$.

**Geometrical Interpretation**



| | |
|---|---|
| **Algorithm:** | **Euler's Method for IVP** |
| **Input:** | (i). The function $f(t, y)$<br>(ii). The end points of the interval $[a, b]$ : $a$ and $b$<br>(iii). The initial value: $\alpha = y(t_0) = y(a)$ |
| **Output:** | Approximations $y_{i+1}$ of $y(t_i + 1)$, $i = 0, 1, \cdots, N-1$. |
| **Step 1.** | **Initialization:** Set $t_0 = a, y_0 = y(t_0) = y(a) = \alpha$. and $N = \frac{b-a}{h}$. |
| **Step 2.** | For $i = 0, 1, \cdots, N-1$ do<br>Compute $y_{i+1} = y_i + hf(t_i, y_i)$<br>End |

**Example:** $y' = t^2 + 5, \qquad 0 \le t \le 1.$

$$y(0) = 0, \; h = 0.25.$$

Find $y_1$, $y_2$, $y_3$, and $y_4$, approximations of $y(0.25)$, $y(0.50$, $y(0.75)$, and $y(1)$, respectively.

The **points of subdivisions** are: $\boxed{t_0 = 0, t_1 = 0.25, t_2 = 0.50, t_3 = 0.75 \text{ and } t_4 = 1.}$

$$\begin{aligned}
\boldsymbol{i=0}: \quad t_1 = t_0 + h = 0.25 & \\
y_1 &= y_0 + hf(t_0, y_0) = 0 + .25(5) = 1.25 \\
(y(0.25) &= 1.2552). \\
\boldsymbol{i=1}: \quad t_2 = t_1 + h = 0.50 & \\
y_2 &= y_1 + hf(t_1, y_1) \\
&= 1.25 + 0.25(t_1^2 + 5) = 1.25 + 0.25((0.25)^2 + 5) \\
&= 2.5156 \\
(y(0.5) &= 2.5417). \\
\boldsymbol{i=2}: \quad t_3 = t_2 + h = 0.75 & \\
y_3 &= y_2 + hf(t_2, y_2) \\
&= 2.5156 + .25((.5)^2 + 5) = 3.8281 \\
(y(0.75) &= 3.8906).
\end{aligned}$$

etc.

**Example:** $y' = t^2 + 5, \;\; 0 \le t \le 2,$

$$y(0) = 0, h = 0.5$$

So, the **points of subdivisions** are: $\boxed{t_0 = 0, \; t_1 = 0.5, \; t_2 = 1, \; t_3 = 1.5, \; t_4 = 2.}$

We compute $y_1, y_2, y_3$, and $y_4$, which are, respectively, approximations to $y(0.5), y(1), y(1.5)$, and $y(2)$.

$$i = 0: \qquad y_1 = y_0 + hf(t_0, y_0) = y(0) + hf(0,0) = 0 + 0.5 \times 5 = 2.5$$

$$(y(0.50) = 2.5417).$$

$$i = 1: \qquad y_2 = y_1 + hf(t_1, y_1) = 2.5 + 0.5((0.5)^2 + 5) = 5.1250$$

$$(y(1) = 5.3333).$$

$$i = 2: \quad y_3 = y_2 + hf(t_2, y_2) = 5.1250 + 0.5(t_2^2 + 5) = 5.1250 + 0.5(1.5) = 8.1250$$

$$(y(1.5) = 8.6250).$$

etc.

**The Errors in Euler's Method**

The approximations obtained by a numerical method to solve the IVP are usually subjected to three types of errors:

- **Local Truncation Error**
- **Global Truncation Error**
- **Round-off Error**

We will not consider the round-off error for our discussions below.

The **local discretization error** is the error made at a **single step** due to the truncation of the series used to solve the problem.

Recall that the Euler Method was obtained by truncating the Taylor series

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots$$

after two terms. Thus, in obtaining Euler's method, the first term neglected is $\boxed{\frac{h^2}{2}y''(t).}$

So the **local error in Euler's method is:** $E_L^E = \frac{h^2}{2}y''(\xi_i)$,

where $\xi_i$ lies between $t_i$ and $t_{i+1}$. In this case, we say **that the local error is of order $h^2$, written as $O(h^2)$.** Note that the local error $E_L^E$ converges to zero as $h \to 0$.

**Global error** is the difference between the true solution $y(t_i)$ and the approximate solution $y_i$ at $t = t_i$. Thus, **Global error** $= y(t_i) - y_i$. Denote this by $E_G^E$.

The following theorem shows that the global error, $E_G^E$, is of order $h$.

**Theorem: (Global Error Bound for the Euler Method)**

Let $y(t)$ be the unique solution of the IVP: $y' = f(t, y); y(a) = \alpha$.

$$a \le t \le b, -\infty < y < \infty,$$

Let $L$ and $M$ be two numbers such that

$$\left|\frac{\partial f(t, y)}{\partial y}\right| \le L, \text{ and } |y''(t)| \le M \text{ in } [a, b].$$

Then the global error $E_G^E$ at $t = t_i$ satisfies

$$|E_G^E| \quad = \quad \left|y(t_i) - y_i\right| \le \frac{hM}{2L}(e^{L(t_i - a)} - 1).$$

**Thus, The global error bound for Euler's method depends upon $h$, whereas the local error depends upon $h^2$.**

Proof of the above theorem can be found in the book by G.W. Gear, **Numerical Initial Value Problems in Ordinary Differential Equations**, Prentice Hall, Inc. (1971).

**Remark.** Since the exact solution $y(t)$ of the IVP is not known, the above bound may not be of practical importance as far as knowing how large the error can be a priori. However, from this error bound, we can say that the *Euler method can be made to converge faster by decreasing the step-size*. Furthermore, if the equalities, $L$ and $M$ of the above theorem can be found, then we can determine what step-size will be needed to achieve a certain accuracy, as the following example shows.

**Example**: $\dfrac{dy}{dt} = \dfrac{t^2 + y^2}{2}, y(0) = 0$

$0 \leq t \leq 1, \ -1 \leq y(t) \leq 1.$

Determine how small the step-size should be so that the error does not exceed $\epsilon = 10^{-4}$.

**Compute $L$:**

Since $f(t, y) = \dfrac{t^2 + y^2}{2}$, we have

$$\frac{\partial f}{\partial y} = y$$

Thus, $\left| \dfrac{\partial f}{\partial y} \right| \leq 1$ for all $y$, giving $\boxed{\text{L=1}}$.

**Find $M$:**

To find $M$, we compute the second-derivative of $y(t)$ as follows:

$$
\begin{aligned}
y' &= \frac{dy}{dt} = f(t, y)(\text{Given}) \\
\text{By implicit differentiation, } y'' &= \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} \\
&= t + \left(\frac{t^2 + y^2}{2}\right) y = t + \frac{y}{2}(t^2 + y^2) \\
\text{So, } |y''(t)| &= |t + \frac{y}{2}(t^2 + y^2)| \leq 2, \text{ for } -1 \leq y \leq 1. \\
\text{Thus, } & \boxed{\text{M=2,}} \\
\text{thus, } \textbf{Global Error Bound:} & \ |E_G^E| \text{ at } t = t_i = |y(t_i) - y_i| \leq \frac{2h}{2L}(e^{(t_i)} - 1) \\
&= h(e^{(t_i)} - 1) = h(e - 1).
\end{aligned}
$$

So, for the error not to exceed $10^{-4}$, we must have:
$h(e-1) < 10^{-4}$ or $h < \dfrac{10^{-4}}{e-1} \approx 5.8198 \times 10^{-5}$.

# 3 High-order Taylor Methods

Recall that the Taylor's series expansion of $y(t)$ of degree $n$ is given by

$$
y(t_{i+1}) \quad = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)
$$

Now,

$$
\begin{aligned}
\text{(i)} \quad & y'(t) = f(t, y(t)) \text{ (Given).} \\
\text{(ii)} \quad & y''(t) = f'(t, y(t)). \\
\text{In general (iii)} \quad & y^{(i)}(t) = f^{(i-1)}(t, y(t)), i = 1, 2, \ldots, n.
\end{aligned}
$$

Thus,

$$y(t_{i+1}) = y(t_i)+hf(t_i,y(t_i))+\frac{h^2}{2}f'(t_i,y(t_i))+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(t_i,y_i)+\frac{h^n}{n!}f^{(n-1)}(t_i,y(t_i))+ \frac{h^{n+1}}{(n+1)!}f^{(n-1)}(\xi_i,y(\xi_i)$$

$$= y(t_i) + h\left[f(t_i,y(t_i)) + \frac{h}{2}f'(t_i,y(t_i)) + \cdots + \frac{h^{n-1}}{n!}f^{n-1}(t_i,y(t_i))\right] + \text{ Remainder Term}$$

Neglecting the remainder term the above formula can be written in compact form as follows:

$y_{i+1} = y_i + hT_k(t_i,y_i),\ i = 0,1,\cdots,N-1$, where $T_k(t_i.y_i)$ is defined by:

$$T_k(t_i,y_1) = f(t_i,y_i) + \frac{h}{2}f'(t_i,y_i) + \cdots + \frac{h^{k-1}}{k!}f^{(k-1)}(t_i,\ y_i)$$

So, if we truncate the Taylor Series after $(k+1)$ terms and use the truncated series to obtain the approximating of $y_{1+1}$ of $y(t_{i+1})$, we have the following **of k-th order Taylor's algorithm for the IVP**.

**Taylor's Algorithm of order $k$ for IVP**

**Input:** (i) The function $f(t, y)$

(ii) The end points: $a$ and $b$

(iii) The initial value: $\alpha = y(t_o) = y(a)$

(iv) The order of the algorithm: $k$

(v) The step size: $h$

**Step 1** **Initialization:** $t_0 = a,\ y_0 = \alpha,\ N = \dfrac{b-a}{h}$

**Step 2.** For $i = \ldots, N-1$ *do*

**2.1** Compute $T_k(t_i, y_i) = f(t_i, y_i) + \dfrac{h}{2} f'(t_i, y_i) + \ldots + \dfrac{h^{k-1}}{k!} f^{(k-1)}(t_i, y_i)$

**2.2** Compute $y_{i+1} = y_i + hT_k(t_i, y_i)$

End

**Note:** With $k = 1$, the above formula for $y_{i+1}$, reduces to Euler's method.

**Example:**

$y' = y - t^2 + 1, \quad 0 \le t \le 2,\ y(0) = 0.5, h = 0 \cdot 2.$

The **points of division** are:

$t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, $t_4 = 0.8$, $t_5 = 1$, and so on.

$$
\begin{aligned}
f(t,y(t)) &= y - t^2 + 1 \text{ (Given).}\\
f'(t,y(t)) &= \frac{d}{dt}(y - t^2 + 1) = y' - 2t\\
&= y - t^2 + 1 - 2t\\
f''(t,y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1\\
\text{so,} & \\
y_1 &= y_0 + hf(t_0, y(t_0)) + \frac{h^2}{2} f'(t_0, y(t_0))\\
&= 0.5 + 0.2 \times 1.5 + \frac{(0.2)^2}{2}(0.5+1) = 0.8300 \text{ (\textbf{approximate value} of } y(0.2)).\\
y_2 &= 1.215800 \text{ (\textbf{approximate value} of } y(0.4)).\\
\text{etc.} &
\end{aligned}
$$

# 4 Runge-Kutta Methods

- The Euler's method is the simplest to implement; however, even for a reasonable accuracy the step-size $h$ needs to be very small.

- The difficulties with higher order Taylor's series methods are that the derivatives of higher orders of $f(t, y)$ need to be computed, which are very often difficult to compute; indeed, $f(t, y)$ is not even explicity known in many areas.

*The Runge-Kutta methods aim at achieving the accuracy of higher order Taylor series methods without computing the higher order derivatives.*

We first develop the simplest one: **The Runge-Kutta Methods of order 2.**

**The Runge-Kutta Methods of order 2**

Suppose that we want an expression of the approximation $y_{i+1}$ in the form:

$$y_{i+1} = y_i + \alpha_1 k_1 + \alpha_2 k_2, \tag{4.1}$$

$$\text{where } k_1 = hf(t_i, y_i), \tag{4.2}$$

and

$$k_2 = hf(t_i + \alpha h,\ y_i + \beta k_1). \tag{4.3}$$

*The constants $\alpha_1$ and $\alpha_2$ and $\alpha$ and $\beta$ are to be chosen so that the formula is as accurate as the Taylor's Series Method of as high as possible.*

To develop the method we need an important result from Calculus: **Taylor's series for function to two variables.**

**Taylor's Theorem for Function of Two Variables**

Let $f(t, y)$ and its partial derivatives of orders up to $(n+1)$ are continuous in the domain $D = \{(t, y) | a \le t \le b,\ c \le y \le d\}$.

Then

$$f(t,y) = f(t_0, y_0) + \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right] + \cdots$$

$$+ \left[\frac{1}{n!}\sum_{h=0}^{n}\binom{n}{i}(t - t_0)^{h-i}(y - y_0)^i \frac{\partial^n f}{\partial t^{n-1}\partial y^i}(t_0, y_0)\right] + R_n(t, y),$$

where $R_n(t, y)$ is the remainder after $n$ terms and involves the partial derivative of order $n + 1$.

Using the above theorem with $n = 1$, we have

$$f(t_i + \alpha h,\ y_i + \beta k_1) = f(t_i, y_i) + \alpha h \frac{\partial f}{\partial t}(t_i, y_i) + \beta k_1 \frac{\partial f}{\partial y}(t_i, y_i) \tag{4.4}$$

From (4.4) and (4.3), we obtain

$$\frac{k_2}{h} = f(t_i, y_i) + \alpha h \frac{\partial f}{\partial t}(t_i.y_i) + \beta k_1 \frac{\partial f}{\partial y}(t_i, y_i). \tag{4.5}$$

Again, substituting the value of $k_1$ from (4.2) and $k_2$ from (4.3) in (4.1) we get (after some rearrangment):

$$\begin{aligned} y_{i+1} &= y_i + \alpha_1 h f(t_i, y_i) + \alpha_2 h \left[ f(t_i, y_i) + \alpha h \frac{\partial f}{\partial t}(t_i, y_i) + \beta h f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \right] \\ &= y_i + (\alpha_1 + \alpha_2) h f(t_i, y_i) + \alpha_2 h^2 \left[ \alpha \frac{\partial f}{\partial t}(t_i, y_i) + \beta f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \right] \end{aligned} \tag{4.6}$$

Also, note that $y(t_{i+1}) = y(t_i) + hf(t_i, y_i) + \frac{h^2}{2}\left( \frac{\partial f}{\partial t}(t_i, y_i) + f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \right)$ + higher order terms.

So, neglecting the higher order terms, we can write

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}\left( \frac{\partial f}{\partial t}(t_i, y_i) + f \frac{\partial f}{\partial y}(t_i, y_i) \right). \tag{4.7}$$

If we want (4.6) and (4.7) to agree for numerical approximations, then we must have

- $\alpha_1 + \alpha_2 = 1$ (comparing the coefficients of $hf(t_i, y_i)$).
- $\alpha_2 \alpha = \frac{1}{2}$ (comparing the coefficients of $h^2 \frac{\partial f}{\partial t}(t_i, y_i)$.
- $\alpha_2 \beta = \frac{1}{2}$ (comparing the coefficents of $h^2 f(t_i, y_i) \frac{\partial f}{\partial y}(t_i y_i)$.

Since the number of unknowns here exceeds the number of equations, there are infinitely many possible solutions. The simplest solution is:

$$\boxed{\alpha_1 = \alpha_2 = \frac{1}{2},\ \alpha = \beta = 1}.$$

With these choices we can generate $y_{i+1}$ from $y_i$ as follows. The process is known as the **Modified Euler's Method**.

**Generating $y_{i+1}$ from $y_i$ in Modified Euler's Method**

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(t_i, y_i)$

$$k_2 = hf(t_i + h,\ y_i + k_1).$$

or

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_i + h,\ y_i + hf(t_i, y_i)\right]$$

**Algorithm: The Modified Euler Method**

**Inputs:** The given function: $f(t, y)$
The end points of the interval: $a$ and $b$
The step-size: $h$
The initial value $y(t_0) = y(a) = \alpha$

**Outputs:** Approximations $y_{i+1}$ of $y(t_{i+1}) = y(t_0 + ih)$,
$i = 0, 1, 2, \cdots, N-1$

**Step 1** (**Initialization**)
Set $t_0 = a,\ y_0 = y(t_0) = y(a) = \alpha$
$N = \dfrac{b-a}{h}$

**Step 2** For $i = 0, 1, 2, \cdots, N-1$ do
Compute $k_1 = hf(t_i, y_i)$
Compute $k_2 = hf(t_i + h, y_i + k_1)$
Compute $y_{i+1} = y_i + \dfrac{1}{2}(k_1 + k_2)$.

End

**Example:**

$y' = e^t$, $y(0) = 1$, $h = 0.5$, $0 \le t \le 1$

$\boxed{t_0 = 0,\ t_1 = 0.5,\ t_2 = 1}$

$$
\begin{aligned}
\mathbf{i = 0}: \qquad k_1 &= hf(t_0, y_0) = 0.5e^{t_0} = 0.5 \\
k_2 &= hf(t_0 + h, y_0 + k_1) = 0.5(e^{t_0+h}) = 0.5e^{0.5} = 0.8244 \\
y_1 &= y_0 + \tfrac{1}{2}(k_1 + k_2) = 1 + 0.5(0.5 + 0.8244) = 1.6622 \\
(y(0.5) &= e^{0.5} = 1.6487) \\
\mathbf{i = 1}: \qquad k_1 &= hf(t_1, y_1) = 0.5e^{t_1} = 0.5e^{0.5} = 0.8244 \\
k_2 &= hf(t_1 + h, y_1 + k_1) = 0.5e^{0.5+0.5} = 0.5e = 1.3591 \\
y_2 &= y_1 + \tfrac{1}{2}(k_1 + k_2) = 1.6622 + \tfrac{1}{2}(0.8244 + 1.3591) = 2.7539 \\
(y(1) &= 2.7183).
\end{aligned}
$$

**Example:** Given: $y' = t + y$, $y(0) = 1$, compute $y_1$ (approximation to $y(0.01)$) and $y_2$ (approximation to $y(0.02)$ by using Modified Euler Method.

$h = 0.01, y_0 = y(0) = 1.$

$$
\begin{aligned}
i = 0: \quad y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\
k_1 &= hf(t_0, y_0) = 0.01(0 + 1) = 0.01 \\
k_2 &= hf(t_0 + h,\ y_0 + k_1) = 0.01 \times f(0.01,\ 1 + 0.01) \\
&\qquad = 0.01 \times (0.01 + 1.01) = 0.01 \times 1.02 = 0.0102
\end{aligned}
$$

Thus $y_1 = 1 + \frac{1}{2}(0.01 + 0.0102) = 1.0101$ (**Approximate value** of $y(0.01)$

$$
\begin{aligned}
i = 1: \quad y_2 &= y_1 + \frac{1}{2}(k_1 + k_2) \\
k_1 &= hf(t_1, y_1) \\
&= 0.01 \times f(0.01, 1.0101) = 0.01 \times (0.01 + 1.0101) \\
&= 0.0102 \\
k_2 &= hf(t_1 + h,\ y_1 + k_1) \\
&= 0.01 \times f(0.02,\ 1.0101 + 0.0102) = 0.01 \times (0.02 + 1.0203) \\
&= -0.0104
\end{aligned}
$$

$y_2 = 1.0101 + \frac{1}{2}(0.0102 + 0.0104) = 1.0204$ (**Approximate value** of $y(0.02)$).

**Local Error in the Modified Euler Method**

Since in deriving the modified Euler method, we neglected the terms involving $h^3$ and higher powers of $h$, the **local error for this method is $O(h^3)$. Thus with the modified Euler method, we will be able to use larger step-size $h$ than the Euler method to obtain the same accuracy**.

## The Midpoint and Heun's Methods

In deriving the modified Euler's Method, we have considered only one set of possible values of $\alpha_1, \alpha_2, \alpha_1$ and $\beta$. We will now consider two more sets of values.

- $\alpha = 0,\ \alpha_2 = 1,\ \alpha = \beta = \dfrac{1}{2}.$

This gives us the **Midpoint Method**.

**The Midpoint Method**

$y_{i+1} = y_i + k_2$

where $k_1 = hf(t_i, y_i)$,

and

$$k_2 = hf\left(t_i + \frac{h}{2},\ \ y_i + \frac{k_1}{2}\right)$$

or

$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2},\ \ y_i + \frac{h}{2} f(t_i, y_i)\right), i = 0, 1, \ldots, N-1.$$

**Example**

$y' = e^t,\ y(0) = 1,\ h = 0.5,\ 0 \le t \le 1.$

$\boxed{t_0 = 0,\ t_1 = 0.5,\ t_2 = 1}$

**Compute $\mathbf{y_1}$**, an approximation to $y(0.5)$:

$$\begin{aligned} \mathbf{i = 0}: \qquad k_1 &= hf(t_0, y_0) = 0.5e^{t_0} = 0.5e^0 = 0.5 \\ k_2 &= hf(t_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}) = 0.5e^{\frac{0.5}{2}} = 0.6420 \\ y_1 &= y_0 + k_2 = 1 + 0.6420 = 1.6420 \\ (y(0.5) &= 1.6487). \end{aligned}$$

**Compute $\mathbf{y_2}$**, an approximation of $y(1)$:

$$\begin{aligned} \mathbf{i = 1}: \qquad k_1 &= hf(t_1, y_1) = 0.5e^{0.5} = 0.8244 \\ k_2 &= hf(t_1 + \tfrac{h}{2}, y_1 + \tfrac{k_1}{2}) = 0.5e^{.75} = 1.0585 \\ y_2 &= y_1 + k_2 = 1.6420 + 1.0585 = 2.7005 \\ (y(1) &= e = 2.7183) \end{aligned}$$

- $\alpha_1 = \frac{1}{4},\ \beta_1 = \frac{3}{4},\ \alpha = \beta = \frac{2}{3}$

Then we have **Heun's Method**.

**Heun's Method**

$$y_{i+1} = y_i + \frac{1}{4}k_1 + \frac{3}{4}k_2$$

where $k_1 = hf(t_i, y_i)$

$$k_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}k_1\right)$$

or

$$y_{i+1} = y_i + \frac{h}{4}f(t_i, y_i) + \frac{3h}{4}f\left(t_i + \frac{2}{3}h, y_i + \frac{2h}{3}f(t_i, y_i)\right), i = 0, 1, \cdots, N-1$$

**Heun's Method** and the **Modified Euler's Method** are classified as the **Runge-Kutta methods of order 2**.

## The Runge-Kutta Method of order 4.

A method widely used in practice is the Runge-Kutta method of order 4. It's derivation is complicated. We will just state the method without proof.

**Algorithm: The Runge-Kutta Method of Order 4**

**Inputs:** $f(t, y)$ - the given function
$a, b$ - the end points of the interval
$\alpha$ - the initial value $y(t_0)$
$h$ - the step size

**Outputs:** The approximations $y_{i+1}$ of $y(t_{i+1})$, $i = 0, 1, \cdots, N-1$

**Step 1**: **(Initialization)**
Set $t_0 = a$, $y_0 = y(t_0) = y(a) = \alpha$
$N = \frac{b-a}{h}$.

**Step 2:** **(Computations of the Runge-Kutta Coefficients)**
For $i = 0, 1, 2, \cdots, n$ do
$k_1 = hf(t_i, y_i)$

$$k_2 = hf(t_i + \frac{h}{2}, \ \ y_i + \frac{1}{2}k_1)$$

$$k_3 = hf(t_i + \frac{h}{2}, \ \ y_i + \frac{1}{2}k_2)$$

$$k_4 = hf(t_i + h, \ \ y_i + k_3)$$

**Step 3:** **(Computation of the Approximate Solution)**
**Compute:** $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

**The Local Truncation Error**: The local truncation error of the Runge-Kutta Method of order 4 is $O(h^5)$.

**Example:**

$$y' = t + y, \ \ y(0) = 1$$

$$h = 0.01$$

Let's complete $y(0.01)$ using the Runge-Kutta Method of order 4.

**i = 0**

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(t_0, y_0) = 0.01f(0,1) = 0.01 \times 1 = 0.01$.

$$k_2 = hf(t_0 + \frac{h}{2},\ \ y_0 + \frac{k_1}{2}) = 0.01f\left(\frac{0.01}{2}, 1 + \frac{0.01}{2}\right) = 0.01\left[\frac{0.01}{2} + \frac{1 + 0.01}{2}\right] = 0.0101.$$

$$k_3 = hf\left(t_0 + \frac{h}{2},\ \ y_0 + \frac{k_2}{2}\right) = h\left(t_0 + \frac{h}{2} + y_0 + \frac{k_2}{2}\right) = 0.0101005.$$

$$k_4 = hf(t_0 + h, y_0 + k_3) = h(t_0 + h + y_0 + k_3) = 0.01020100$$

So, $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.010100334$

and so on.