

2. (a) Show that the Chebyshev polynomial $T_k(x)$ of degree $k \geq 1$ has k simple zeros in the interval $[-1, 1]$ at

$$x_j = \cos\left(\frac{2j-1}{2k}\pi\right), \quad j = 1, 2, \dots, k$$

and $T_k(x)$ has extreme points at $z_j = \cos\left(\frac{j\pi}{k}\right)$, $j = 0, 1, \dots, k$.

- (b) What are the optimal nodes for interpolating with a polynomial of degree 2 on $[2, 5]$?

- (c) Using the zeros of $\tilde{T}_3(x)$, construct an interpolating polynomial of degree 2 for $f(x) = e^x$ on $[2, 5]$.

3. (a) Consider the Chebyshev least-squares approximation of $f(x)$ on $[-1, 1]$ in the form

$$P_m(x) = \sum_{j=0}^m a_j T_j(x)$$

Find the expressions for a_j in terms of $T_j(x)$ and $f(x)$.

- (b) Show that for $f(x) = \arccos x$ on $[-1, 1]$, the expression for a_j in 3(a) is given by

$$a_j = \begin{cases} -\frac{4}{\pi} j^2 & \text{when } j \text{ is odd} \\ 0 & \text{when } j \text{ even.} \end{cases}$$

4. Using the 4th degree polynomial $p_4(x)$ for $\sin x$ (obtained from a power series expansion), find a polynomial $p_3(x)$ of degree 3 such that

$$\max_{-1 \leq x \leq 1} |p_4(x) - p_3(x)|$$

will be minimum.

What is the total error (sum of the truncation error + error by Cheysheer approximation)?