

1. Learn the derivations of the composite Trapezoidal and Simpson's rules and those of their error formulas.

$$2. \int_{x_0}^{x_2} f(x) dx = a_0 f_0 + a_1 f_1 + a_2 f_2 + K f^{(4)}(\eta)$$

$$f(x) = 1$$

$$\int_{x_0}^{x_2} 1 dx = x_2 - x_0 = a_0 \cdot 1 + a_1 \cdot 1 + a_2 \cdot 1$$

$$f(x) = x$$

$$\int_{x_0}^{x_2} x dx = \frac{(x_2 - x_0)^2}{2} = a_0 x_0 + a_1 \left(\frac{x_0 + x_2}{2}\right)^2 + a_2 x_2$$

$$f(x) = x^2$$

$$\int_{x_0}^{x_2} x^2 dx = \frac{(x_2 - x_0)^3}{3} = a_0 x_0^2 + a_1 \left(\frac{x_0 + x_2}{2}\right)^2 + a_2 x_2^2$$

$$\Rightarrow a_0 = \frac{h}{3}, a_1 = \frac{4}{3}h, a_2 = \frac{h}{3},$$

$$\text{Where } h = \frac{x_2 - x_0}{2}.$$

Let  $f(x) = x^3$ , we find that

$$\int_{x_0}^{x_2} x^3 dx = \frac{(x_2 - x_0)^4}{4} = \frac{x_2 - x_0}{6} x_0^3 + \frac{2(x_2 - x_0)}{3} \left(\frac{x_0 + x_2}{2}\right)^3 + \frac{x_2 - x_0}{6} x_2^3$$

Let  $f(x) = x^4$ ,

$$\Rightarrow K = -\frac{\left(\frac{x_2 - x_0}{2}\right)^5}{90}.$$

$$3. |E_{CT}| = \left| \frac{b-a}{12} h^2 f''(\eta) \right| < \varepsilon$$

$$\Rightarrow h < \sqrt{\frac{\varepsilon \cdot 12}{(b-a) \cdot \max\{f''(\eta)\}}}$$

$$|E_{CS}| = \left| -\frac{(b-a)}{180} h^4 f^{(4)}(\eta) \right| < \varepsilon$$

$$\Rightarrow h < \frac{\varepsilon \cdot 180}{(b-a) \cdot \max\{f^{(4)}(\eta)\}}$$

For example, (a)  $f(x) = \frac{1}{(x+4)}$ ,  $f''(x) = \frac{2}{(x+4)^3}$ ,  $f^{(4)}(x) = \frac{24}{(x+4)^5}$

For composite Trapezoidal Rule,

$$h < \sqrt{\frac{\varepsilon \cdot 12}{(b-a) \cdot \max\{f''(\eta)\}}} = \sqrt{\frac{10^{-5} \cdot 12}{(2-0) \cdot \frac{2}{(0+4)^3}}} = 0.0438178$$

For Composite Simpson's Rule,

$$h < \left[ \frac{\varepsilon \cdot 180}{(b-a) \cdot \max\{f^{(4)}(\eta)\}} \right]^{\frac{1}{4}} = \left[ \frac{10^{-5} \cdot 180}{(2-0) \cdot \frac{24}{(0+4)^5}} \right]^{\frac{1}{4}} = 0.313016$$

$$4. \int_0^1 x^{\frac{1}{3}} dx = 0.75$$

$$|R_{10,10} - R_{9,9}| = 8.8543 \times 10^{-5} < 10^{-4}$$

$$5. R_{22} = R_{21} + \frac{R_{21} - R_{11}}{4-1}$$

$$\Rightarrow R_{21} = \frac{3}{4} R_{22} + \frac{R_{11}}{4} = \frac{3}{4} \cdot 0.2315 + \frac{0.25}{4} = 0.0109$$

$$6. f(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1-x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

(a) Trapezoidal Rule over [0,1]:

$$\int_0^1 f(x) dx = \frac{1}{2} (f(0) + f(1)) = 0$$

(b) Trapezoidal Rule over [0,1/2] and [1/2,1]:

$$\int_0^1 f(x) dx = \frac{1}{4} (f(0) + 2f(\frac{1}{2}) + f(1)) = \frac{1}{4}$$

(c) Simpson's rule over [0,1]:

$$\int_0^1 f(x) dx = \frac{1/2}{3} (f(0) + 4f(\frac{1}{2}) + f(1)) = \frac{1}{3}$$

7. Learn the derivation of Gaussian Quadrature formula with n=2.

8. Let  $f(x) = 1$ ,  $f'(x) = 0$

$$\int_{-1}^1 1 dx = 2 = \alpha + \beta$$

Let  $f(x) = x, f'(x) = 1$

$$\int_{-1}^1 x dx = 0 = -\alpha + \beta + \nu + \delta$$

Let  $f(x) = x^2, f'(x) = 2x$

$$\int_{-1}^1 x^2 dx = 1 = \alpha + \beta - 2\nu + 2\delta$$

Let  $f(x) = x^3, f'(x) = 3x^2$

$$\int_{-1}^1 x^3 dx = 0 = -\alpha + \beta + 3\nu + 3\delta$$

$$\Rightarrow \alpha = 1, \beta = 1, \nu = \frac{1}{4}, \delta = -\frac{1}{4}.$$

9.  $x_{1,2} = \pm\sqrt{1/3}, w=1,$

$$\int_a^b f(x) dx = \frac{b-a}{2} \sum_{i=1}^2 w_i \cdot f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right)$$

(a)  $\int_0^{\frac{\pi}{4}} e^{3x} \sin 2x dx \approx 2.5913$

10. Learn all the properties of Legendre and Chebyshev polynomials.

11.  $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{24}$  in  $[-1, 1]$ .

$$\min\{\max_{-1 \leq x \leq 1} |P_4(x) - P_3(x)|\} = \frac{|a_4|}{2^{4-1}} = \frac{1/24}{2^3} = 5.2083 \times 10^{-3}.$$

12. The three zeros of  $\tilde{T}_3(x)$  in  $[-1, 1]$  are given by

$$\tilde{x}_1 = \cos \frac{\pi}{6}, \tilde{x}_2 = \cos \frac{\pi}{2}, \tilde{x}_3 = \cos \frac{5}{6} \pi.$$

These zeros are to be shifted using transformation:

$$x_{new} = \frac{1}{2}[(3-2)\tilde{x}_i + (3+2)].$$

$$\text{So, } \tilde{x}_{1,new} = \frac{\sqrt{3}}{2} + \frac{5}{2}, \tilde{x}_{2,new} = \frac{5}{2}, \tilde{x}_{3,new} = -\frac{\sqrt{3}}{2} + \frac{5}{2}.$$

Those are the best possible choices of interpolation of a function with a polynomial of degree at most 3 in [2,3].

13. Learn the Normal Equation Method for least squares approximations of a set of discrete data and a function in an arbitrary interval.

$$14. \tilde{x}_1 = \cos \frac{\pi}{6}, \tilde{x}_2 = \cos \frac{\pi}{2}, \tilde{x}_3 = \cos \frac{5}{6} \pi,$$

$$\tilde{x}_{1,new} = \frac{\sqrt{3}}{2} + \frac{5}{2}, \tilde{x}_{2,new} = \frac{5}{2}, \tilde{x}_{3,new} = -\frac{\sqrt{3}}{2} + \frac{5}{2},$$

$$\text{For } f(x) = \frac{1}{x}$$

$$a_0 = \frac{1}{\pi} \int_{-1}^1 \frac{dx}{x\sqrt{1-x^2}}, \quad a_1 = \frac{1}{\pi} \int_{-1}^1 \frac{x \cdot dx}{x\sqrt{1-x^2}}, \quad a_2 = \frac{1}{\pi} \int_{-1}^1 \frac{(2x^2-1) \cdot dx}{x\sqrt{1-x^2}}, \quad a_3 = \frac{1}{\pi} \int_{-1}^1 \frac{(4x^3-3x) \cdot dx}{x\sqrt{1-x^2}},$$

$$f(x) = \frac{1}{x} \approx P_3(x) = a_0 + a_1 \tilde{T}_1(x) + a_2 \tilde{T}_2(x) + a_3 \tilde{T}_3(x).$$

$$15. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},$$

$$P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x),$$

$$P_6(x) = P_7(x) - \left(-\frac{1}{7!}\right) \tilde{T}_6,$$

$$E_{6,7} = \frac{|a_7|}{2^6},$$

$$P_5(x) = P_6(x) - \left(-\frac{1}{7!}\right) \tilde{T}_5,$$

$$E_{5,7} = \frac{|a_7|}{2^6} + \frac{|a_6|}{2^5},$$

...

Eventually, we are able to have a polynomial of order 2 to economize the power series of  $\sin x$  while keeping the error less than 0.01.

16. Learn the notes of IVP.

17. Euler's Method:

$$y_{i+1} = y_i + hf(t_i, y_i), i = 0, 1, 2, \dots, N-1$$

$$y_0 = y(0) = 0.5, h = 0.5$$

$$\text{So, } y_1 = y_0 + 0.5 \cdot (y_0 - 0^2 + 1) = 1.75$$

$$y_2 = y_1 + 0.5 \cdot (y_1 - (0.5)^2 + 1) = 3$$

$$y_3 = y_2 + 0.5 \cdot (y_2 - 1^2 + 1) = 4.5$$

$$y_4 = y_3 + 0.5 \cdot (y_3 - (1.5)^2 + 1) = 6.125$$

$$E_L^E = \frac{h^2}{2} y''(\xi_i) = \frac{(0.5)^2}{2} (-2) = -0.25$$

$$|E_G^E| \leq \frac{hM}{2L} (e^{L(t_i-1)} - 1), \text{ where } \left| \frac{\partial f(t, y)}{\partial y} \right| \leq L = 0.5, |y''(t)| \leq M = 2,$$

$$\text{So, } |E_G^E| \leq \frac{hM}{2L} (e^{L(t_i-1)} - 1) = 0.64872127.$$

18. See 17.

$$19. y' = f(t, y) = -y + t + 1,$$

Mid Point Method:

$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right), i = 0, 1, \dots, N-1,$$

Modified Euler's Method:

$$y_{i+1} = y_i + \frac{h}{2} [f(t_i, y_i) + f(t_i + h, y_i + hf(t_i, y_i))],$$

Heun's Method:

$$y_{i+1} = y_i + \frac{h}{4} f(t_i, y_i) + \frac{3h}{4} f\left(t_i + \frac{2}{3}h, y_i + \frac{2h}{3} f(t_i, y_i)\right), i = 0, 1, \dots, N-1.$$