Solution of EXAM #2

1. (a) Derive the composite Trapezoidal rule.

- Divide the interval $[a, b]$ into n equal subintervals.
- Integrate $f(x)$ over each of those subintervals using the Trapezoidal rule for each subinterval.
- Add the results.

$$
\begin{aligned}
I_{CT} &= \int_{x_0=a}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_{n=b}} f(x)dx \\
&= \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \ldots + \frac{h}{2}(f_{n-1} + f_n) \\
&= h\left(\frac{f_0}{2} + f_1 + f_2 + \ldots + f_{n-1} + \frac{f_n}{2}\right).
\end{aligned}
$$

So, the composite trapezoidal Rule is:

$$
I_{CT} = h\left[\frac{f_0}{2} + f_1 + f_2 + \ldots + f_{n-1} + \frac{f_n}{2}\right]
$$

$$
= \frac{h}{2}\left[f(a) + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f(b)\right].
$$

The error of composite trapezoidal Rule:

$$
\begin{aligned}
E_{CT} &= -\frac{nh^3}{12} f''(\eta), \text{ where } \eta_1 < \eta < \eta_n. \\
&= -n\frac{(b-a)}{n} \cdot \frac{h^2}{12} f''(\eta) = -\frac{(b-a)}{12} h^2 f''(\eta),
\end{aligned}
$$

where $a < \eta < b$.

(b) Determine h so that the composite Trapezoidal rule gives the value of

$$\int_0^1 e^{-x^2} dx$$

With an accuracy of $\varepsilon = 10^{-7}$.

$$\frac{(1-0)}{12} h^2 f''(\eta) < \varepsilon$$

$$| f''(\eta) | = | (e^{-x^2})'' | = | (-2 + 4x^2) \cdot e^{x^2} | \le 2$$

$h^2 < \dfrac{12 \cdot \varepsilon}{\max\{f''(\eta)\}} = \dfrac{12 \cdot 10^{-7}}{2} = 6 \times 10^{-7}$, so $h < 7.74597 \times 10^{-4}$.

2. Approximate $\int_1^{1.5} e^{-x^2}\,dx$ using Gaussian Quadrature with n=2.

Sol: Changing of variables: $x = \frac{1}{2}[(1.5-1)t + (1+1.5)] = \frac{1}{4}t + \frac{5}{4}$

So, $dx = \frac{1}{4}dt$

$$\int_1^{1.5} e^{-x^2}\,dx = \frac{1}{4}\int_{-1}^{1} e^{-(\frac{t}{4}+\frac{5}{4})^2}\,dt = a_1 f(t_1) + a_2 f(t_2)$$

Where $f(t) = \frac{1}{4}e^{-(\frac{t}{4}+\frac{5}{4})^2}$

For $n = 2$

$a_1 = a_2 = 1$,

and the roots of the Legendre polynomial of degree 2 : $t_{1,2} = \pm\frac{1}{\sqrt{3}}$

So, $\int_1^{1.5} e^{-x^2}\,dx = a_1 f(t_1) + a_2 f(t_2) = \frac{1}{4}e^{-(\frac{1}{4\sqrt{3}}+\frac{5}{4})^2} + \frac{1}{4}e^{-(\frac{1}{4\sqrt{3}}+\frac{5}{4})^2}$.

(b) Construct a quadrature rule of the form:

$$\int_{-1}^{1} f(x)dx \approx A_0(-\frac{1}{2}) + A_1 f(0) + A_2 f(\frac{1}{2})$$

Which is exact for all polynomial of degree ≤ 2 .

$f(x) = 1$

$$\int_{-1}^{1} 1dx = 2 = A_0 + A_1 + A_2$$

$f(x) = x$

$$\int_{-1}^{1} xdx = 0 = -\frac{1}{2}A_0 + \frac{1}{2}A_2$$

$f(x) = x^2$

$$\int_{-1}^{1} x^2dx = \frac{2}{3} = \frac{1}{4}A_0 + \frac{1}{4}A_2$$

3. Determine an upper bound for the global error at t=1 of Euler's Method in solving $y' = y$, $y(0) = 1$ from $t = 0$, to $t = 1$, $h = 0.5$. (Note that the exact solution is $y(t) = e^t$).

Sol: The global error $|E_G^E| = |y(t_i) - y_i| \le \dfrac{hM}{2L}(e^{L(t_i - a)} - 1)$

Where $L \ge |\dfrac{\partial f(t,y)}{\partial y}| = 1$, $|y''(t)| = |e^t| \le M = e$, $h = 0.5$, $t_1 = 0.5, t_2 = 1$

$|E_G^E| \le \dfrac{hM}{2L}(e^{L(t_i - a)} - 1) = \dfrac{0.5 \cdot e}{2}(e - 1) = 1.16769$.

4. (a) Given $P_3(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}$, find a polynomial $P_2(x)$ of degree 2 such that $\max_{-1\le x\le 1}|P_3(x)-P_2(x)|$ is minimized. What is the minimum value?

Sol: $P_2(x)=P_3(x)-a_3\tilde{T}_3(x)$,

$T_3(x)=4x^3-3x \Rightarrow \tilde{T}_3(x)=x^3-\frac{3}{4}x$,

So, $P_2(x)=P_3(x)-a_3\tilde{T}_3(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{1}{6}(x^3-\frac{3}{4}x)=1+\frac{9}{8}x+\frac{x^2}{2}$.

(b) Use Legendre polynomial of degree 2 to approximate $f(x)=x^3$ on the $[-1,1]$.

Sol: $P(x)=a_0\Phi_0(x)+a_1\Phi_1(x)+a_2\Phi_2(x)$,

Where $\Phi_0(x)$, $\Phi_1(x)$, $\Phi_2(x)$ are the legendre polynomial of degree 0, 1 and 2, respectively. And, $a_i=\frac{\int_{-1}^{1}\Phi_i(x)f(x)dx}{\int_{-1}^{1}\Phi_i^2(x)dx}$.

$\Phi_0(x)=1$, $\Phi_1(x)=x$, $\Phi_2(x)=x^2-\frac{1}{3}$,

So, $a_0=\frac{\int_{-1}^{1}x^3dx}{\int_{-1}^{1}dx}$, $a_1=\frac{\int_{-1}^{1}x\cdot x^3dx}{\int_{-1}^{1}x^2dx}$, $a_2=\frac{\int_{-1}^{1}(x^2-\frac{1}{3})\cdot x^3dx}{\int_{-1}^{1}(x^2-\frac{1}{3})^2dx}$.

5. Bonus Problem

Find the best possible interpolating nodes for interpolation of $f(x) = xe^x$ in $[0,1.5]$ using a polynomial of degree at most 2.

Sol: The roots of the polynomial $T_3(x) = 4x^3 - 3x$ are $x_{1,2,3} = 0, \pm\frac{\sqrt{3}}{2}$.

Changing of nodes: $\tilde{x}_i = \frac{1.5-0}{2} x_i + \frac{0+1.5}{2} = \frac{3}{4} x_i + \frac{3}{4}$,

So, $\tilde{x}_1 = 0.75$, $\tilde{x}_2 = 1.39952$, $\tilde{x}_3 = 0.10049$ are the best possible interpolating nodes for interpolation of $f(x) = xe^x$ in $[0,1.5]$ using a polynomial of degree at most 2.