A New Algorithm for Generalized Sylvester-Observer Equation and its Application to State and Velocity Estimations in Vibrating Systems

João B Carvalho [*], email: carvalho@mat.ufrgs.br
Biswa N Datta [†], IEEE Fellow, email: dattab@math.niu.edu

Abstract

We propose a new algorithm for block-wise solution of the generalized Sylvester-observer equation $XA - FXE = GC$, where the matrices A, E, and C are given, the matrices X, F, and G need to be computed, and the matrix E may be singular. The algorithm is based on an orthogonal decomposition of the triplet (A, E, C) to the observer-Hessenberg-triangular form. It is a natural generalization of the widely-known observer-Hessenberg algorithm for the Sylvester-observer equation : $XA - FX = GC$, which arises in state estimation of a standard first-order state-space control system. An application of the proposed algorithm is made to state and velocity estimations of second-order control systems modeling a wide variety of vibrating structures. For dense un-structured data, the proposed algorithm is more efficient than the recently proposed SVD-based algorithm of the authors, it is heavily composed of BLAS-3 (Level 3 Basic Linear Algebra Subprograms) operations, which make the algorithm an ideal candidate for high-performance computing.

Keywords: state estimation, Sylvester-observer equation, block-wise solution

1 Introduction

The Sylvester matrix equation

$$XA - FX = G \tag{1}$$

[*]Departamento de Matemática Pura e Aplicada, Universidade Federal do RS, Brazil.
[†]Dept.Math.Sciences, Northern Illinois University, DeKalb, IL 60115. Author partially supported by NSF grant DMS-0505784, URL: www.math.niu.edu:/∼ dattab

is well-known in both mathematics and control literature. In this equation, the matrices A, F and G are given and the matrix X needs to be determined. An important variation of the Sylvester equation, called the Sylvester-observer equation:

$$XA - FX = GC \tag{2}$$

where the matrices A and C are given and X, F, and G need to be determined, arises in state-estimation of linear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y &= Cx(t) \end{aligned} \tag{3}$$

(see chapter 12 of [**?**] for details). We will assume here the most practical situation that a part of the states (say r components of the vector $x(t)$) have been measured and the remaining $(n - r)$ components need to be determined. This is the case of a *reduced order observer*. The dimension of the matrices in this case, are, respectively, $A \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{(n-r)\times n}$, $F \in \mathbb{R}^{(n-r)\times(n-r)}$, $G \in \mathbb{R}^{(n-r)\times r}$, $C \in \mathbb{R}^{r\times n}$. We also assume, without any loss of generality, that C has full rank; that is $\text{rank}(C) = r$. Equation (**??**) also arises in solutions of related problems of eigenvalue and partial eigenvalue assignments [**?**, **?**, **?**] and in the study of sensitivity analysis of these problems [**?**, **?**, **?**].

There now exist several numerically viable algorithms for solving these matrix equations. They include (i) the well-known Hessenberg-Schur algorithm [**?**] for the Sylvester equation, the Hessenberg-observer algorithm by Van Dooren [**?**], the Hessenberg-observer algorithm by Carvalho and Datta [**?**], the SVD-based algorithm by Datta and Sarkissian[**?**], the QR-based algorithm by Carvalho, Datta, and Hong [**?**]. There also exist large-scale and parallel algorithms [**?**, **?**, **?**] for the Sylvester-observer equation. The paper by Bischop, Datta, Purkayastha[**?**] has developed a parallel algorithm and the paper by Calvetti, Lewis, and Reichel [**?**] has proposed several important modifications of the the algorithm proposed by Datta and Saad [**?**] for large-scale solution of the problem.

A generalized version of the Sylvester-observer equation, to be called the *Generalized Sylvester-observer equation* or descriptor Sylvester-observer equation:

$$XA - FXE = GC \tag{4}$$

where the matrices A, E, and C are given, and X, F, and G need to be determined, naturally arises [**?**, **?**, **?**] in state and velocity estimation of systems of the form:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + \tilde{B}u(t) \\ y(t) &= Cx(t), \end{aligned} \tag{5}$$

called *descriptor systems*. Note that if the matrix E is non-singular, then this equation reduces to the Sylvester-observer equation (**??**).

For the generalized Sylvester-observer equation (**??**), only one block-wise solution has been proposed so far : the recent Singular Value Decomposition (SVD) based method [**?**]. Although this SVD method is numerically attractive, it is quite expensive, since it not only requires numerical solutions of several different Sylvester equations of the type $YA - FYE = J$, but also SVD needs to be performed on each of those solution matrices (Steps 5 and 6 of Algorithm 3.1 in [**?**]), and it is well-known [**?**] that computing the SVD of a matrix is rather an expensive process.

In this paper, we propose another conceptually simple algorithm for solving (**??**). The algorithm is based on the observer-Hessenberg-triangular decomposition of the triple (A, E, C) (which is one of the orthogonal state-space reductions proposed in [**?**] and implemented in [**?**]). Note that it is only natural to develop such an algorithm for this problem because the observer-Hessenberg-triangular form of (A, E, C) immediately generalizes the well-known observer-Hessenberg form of (A, C), which has been an attractive tool for numerically solving major control problems [**?**]. Thus, this new algorithm is a natural generalization of the widely-known algorithm of Van Dooren [**?**] for solving (**??**). The results of numerical experiments on benchmark examples show that *the proposed algorithm is more efficient than the SVD algorithm* when matrices are non-structured. Bedside the observer-Hessenberg-triangular decomposition, the only other major computational requirement of this new algorithm is solving several low-order algebraic linear systems with multiple right hand sides, which really make the algorithm quite attractive for high-performance computing. Since the reduction to observer-Hessenberg-triangular form is achieved by orthogonal equivalence [**?**], *the algorithm should have the same numerical effectiveness as of the observer-Hessenberg algorithm* for solving (**??**).

The following Lemma generalizes the classical results in [**?**].

Lemma ??.1: Assume that the system (**??**) is observable and C has full rank r. If the matrix triplet (X, F, G), where $X \in \mathbb{R}^{(n-r)\times n}$, $F \in \mathbb{R}^{(n-r)\times(n-r)}$, $G \in \mathbb{R}^{(n-r)\times r}$ are computed as a solution of (**??**), with the additional property that (F, G) is controllable and $F \in \mathbb{R}^{(n-r)\times(n-r)}$ is a stable matrix (i.e, all eigenvalues of F have negative real parts), then

$$z(t) - XEx(t) \to 0 \text{ as } \to \infty, \tag{6}$$

where $z(t)$ is the state variable of the controllable *observer system*, governed by

$$\dot{z}(t) = Fz(t) + Gy(t) + X\tilde{B}u(t)\ ,\ z(0) = 0. \tag{7}$$

Proof of Lemma: Define $e(t) = z(t) - XEx(t)$. Then

$$\begin{aligned}\dot{e}(t) = \dot{z}(t) - XE\dot{x}(t) = Fz(t) + GCx(t) + X\tilde{B}u(t) - X(Ax(t) + \tilde{B}u(t)) = \\ = Fz(t) - FXEx(t) = Fe(t).\end{aligned} \tag{8}$$

Since F is assumed to be a stable matrix, it follows that $e(t) \to 0$ as $t \to \infty$, and the result follows. □

As a consequence of Lemma **??**.1, an estimate $\hat{x}(t)$ to the state vector $x(t)$ can be computed by solving

$$\begin{bmatrix} XE \\ C \end{bmatrix} \hat{x}(t) = \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} \tag{9}$$

as long as the computed matrix X has full rank $n - r$ (will be shown in Section 2) and rank$(E) \geq n - r$.

2 A Observer-Hessenberg-triangular Method for Generalized Sylvester-observer Equation

Let $A, E \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{r\times n}$, be such that the system (**??**) is observable and C has full rank r. Let the matrices A and E be reduced to block Hessenberg-triangular form via an orthogonal equivalence transformation. That is, orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ are computed such that

$$\begin{array}{l} U^T AV = H \\ U^T EV = T \\ CV = \tilde{C} = \left[\begin{array}{ccccc} 0 & 0 & \dots & 0 & C_1 \end{array}\right] \end{array} \tag{10}$$

where (i) H is a block upper Hessenberg matrix with blocks $H_{i,j} \in \mathbb{R}^{n_i\times n_j}; i, j = 1, \dots p$;

(ii) The sub-diagonal blocks $H_{i+1,i}$, $i = 1, \dots , p-1$ have full column rank;

(iii) $n_1 \le n_2 \le \dots \le n_p = r$ and $n_1 + n_2 + \dots + n_p = n$;

(iv) T is an upper triangular matrix;

(v) C_1 has full column rank r.

An algorithmic description for the block-wise computation of the matrices U and V in (**??**) can be found in the work of Varga [**?**].

Using (**??**) in (**??**) , and defining $Y = XU$, we obtain

$$YH - FYT = \left[\begin{array}{ccccc} 0 & 0 & \dots & 0 & GC_1 \end{array}\right]. \tag{11}$$

The equation (**??**) is now solved, obtaining Y, F, and G with the additional properties that F is stable and (F, G) is controllable. Finally, the matrix X is recovered from Y as

$$X = YU^T. \tag{12}$$

Development of the Method for Solving (??):

The orthogonal state-space reduction (**??**) gives

$$H = \left[\begin{array}{cccc} H_{11} & H_{12} & \dots & H_{1p} \\ H_{21} & H_{22} & \dots & H_{2p} \\ & H_{32} & \dots & H_{3p} \\ & & H_{p,p-1} & H_{pp} \end{array}\right], \ T = \left[\begin{array}{cccc} T_{11} & T_{12} & \dots & T_{1p} \\ & T_{22} & \dots & T_{2p} \\ & & \ddots & \dots \\ & & & T_{pp} \end{array}\right] \tag{13}$$
$$\tilde{C} = \left[\begin{array}{ccccc} 0 & 0 & \dots & 0 & C_1 \end{array}\right].$$

Let F, Y, and G be conformably partitioned as

$$F = \left[\begin{array}{cccc} F_{11} & & & \\ F_{21} & F_{22} & & \\ & \ddots & \ddots & \\ & & F_{q,q-1} & F_{qq} \end{array}\right], \ Y = \left[\begin{array}{cccc} Y_{11} & Y_{12} & \dots & Y_{1p} \\ & Y_{22} & \dots & Y_{2p} \\ & & Y_{qq} & Y_{qp} \end{array}\right], \ G = \left[\begin{array}{c} G_1 \\ \dots \\ G_q \end{array}\right], \tag{14}$$

where either $q = p$ or $q = p - 1$.

For $i = 1, \ldots, p$, setting the diagonal blocks $Y_{ii} \in \mathbb{R}^{n_i \times n_i}$ as arbitrary non-singular matrices of appropriate orders imply that matrix Y, once constructed, is sure to have full rank.

The blocks F_{ii} of F are chosen to be the quasi-diagonal matrices (having either 2×2 *Schur bumps*, corresponding to a pair of complex conjugate eigenvalues, or real scalars, corresponding to real eigenvalue) that one wants to assign to the observer system. Of course, particular choices can be made to ensure that F remains a stable matrix, as required for the construction of the observer.

Substituting (**??**) and (**??**) into (**??**) and comparing the corresponding blocks on left- and right-hand sides, we obtain a set of equations which is then rearranged and broken down into three sets of linear equations, for finding the unknown blocks of the matrices F, Y and G, as follows:

- For $i = 2, \ldots, q$

$$F_{i,i-1} T_{i-1,i-1} = H_{i,i-1} \tag{15}$$

- For $i = 1, 2, \ldots, q$ and $j = i, \ldots, p-1$

$$Y_{i,j+1} H_{j+1,j} = -\sum_{k=i}^{j} Y_{ik} H_{kj} + \sum_{k=\max(i-1,1)}^{i} F_{ik} \sum_{\ell=k}^{j} Y_{k\ell} T_{\ell j} \tag{16}$$

- For $i = 1, 2, \ldots, q$

$$G_i C_1 = \sum_{k=i}^{p} Y_{ik} H_{kp} - \sum_{k=\max(i-1,1)}^{i} F_{ik} \sum_{\ell=k}^{p} Y_{k\ell} T_{\ell p}. \tag{17}$$

Equation (**??**) is a linear system which is guaranteed to have solution since its coefficient matrix $H_{j+1,j}$ has a full column rank, for $j = 1, 2, \ldots, q$. Therefore, as long as (**??**) has $n - r$ observed states, reduction (**??**) can be used to construct solution matrices (X, F, G) to (**??**) as we are proposing. Moreover, since matrix Y will have full rank by construction, matrix X defined in (**??**) will also have full rank.

Since it is assumed that rank$(E) = n - r$, from the strategy [**?**] of the reduction process(**??**), it follows that infinite controllable states of (**??**), if exist, would be represented in the lower parts of the matrices H and T, and therefore it follows that $T_{pp} \in \mathbb{R}^{r \times r}$ is the only diagonal block of T that could be singular. Therefore, the reduction to observable staircase form (**??**) gives, for $i = 2, 3, \ldots, q$, matrices $H_{i,i-1}$ and $T_{i-1,i-1}$ which have full rank. This also implies that, from (**??**), for $i = 2, 3, \ldots q$, block $F_{i,i-1}$ has full rank.

The above discussion leads to the following algorithm:

Algorithm ??.1: The Observer-Hessenberg-Triangular Algorithm for the Generalized Sylvester-Observer Equation

Input: Matrices $A, E \in R^{n \times n}$, and $C \in \mathbb{R}^{r \times n}$ of the linear system (**??**) .
Output: Block matrices X, F, and G, such that F is stable and $XA - FXE = GC$.
Assumptions: The system (**??**) is observable, and C has full rank. Also, either $s = n$ or $s = n - r$.

Step 1: Reduce (A, E, C) to the observer-Hessenberg- triangular form $(H, T, \tilde{C})$. Let U and V be the orthogonal matrices of this reduction and let $n_i, i = 1, \ldots, p$ be the dimension of the diagonal blocks of the matrix H.
Step 2: Partition the matrices Y, F, and G in blocks conformably with the block structure of H. Set $q = p$ if $s = n$; otherwise, set $q = p - 1$. Let the matrix F be partitioned in blocks according to the partitioning of the top leftmost $s \times s$ sub-matrix of matrix H.
Step 3: For $i = 1, 2, \ldots, q$, define diagonal block $F_{ii} \in \mathbb{R}^{n_i \times n_i}$ to be a stable matrix with eigenvalues disjoint from those of the pair (A, E).
Step 4: Set $Y_{11} = I_{n_1 \times n_1}$.
Step 5: For $i = 2, 3, \ldots, q$, set $Y_{ii} = I_{n_i \times n_i}$ and solve equation (**??**) for $F_{i,i-1} \in \mathbb{R}^{n_i \times n_{i-1}}$.
Step 6: For $i = 1, 2, \ldots, q$, do Steps 7 and 8.
  Step 7: For $j = i, i + 1, \ldots, p - 1$, solve the linear system (**??**) for $Y_{i,j+1}$.
  Step 8: Solve the linear system (**??**) for G_i.
Step 9: Form the block matrices Y, F, and G from their computed blocks.
Step 10: Compute $X = YU^T$.

Important remarks on this algorithm:

1. If the above simple choices for non-singular diagonal blocks $Y_{ii}, i = 1, 2, \ldots, q$ do not construct a controllable pair (F, G), then other choices can be considered to achieve this goal.
2. Properties related to the observability of system (**??**), through the orthogonal reduction (**??**), can possibly reflect on ill-conditioning of sub-diagonal blocks $H_{j+1,j}$, and therefore affect the performance of this algorithm.

Efficiency by Flop-count:

For simplicity, we will assume that $n_1 = n_2 = \ldots = n_p = r$ where $n = p \cdot r$. We will also assume that $q \approx p$, since $q = p - 1$ in this algorithm.

- Step 1, using routine TGO1ID (or TG01AD) of the library SLICOT [**?**], requires $\frac{10n^3}{3}$ flops;
- Step 5 requires about nr^2 flops, taking advantage of the triangular nature of the systems;
- Step 7 requires

$$3(2r^3 + r^2)\sum_{i=1}^{p-1}\sum_{j=i}^{p-1}(j - i + 1) + 2(2r^3 + r^2)\sum_{i=1}^{p-1}\sum_{j=i}^{p-1}(1) \approx 2n^3$$

  flops;
- Step 8 requires about

$$3(2r^3 + r^2)\sum_{i=1}^{p-1}(p - i + 1) + 2(2r^3 + r^2)\sum_{i=1}^{p-1}(1) \approx 3n^2 r$$

flops;

- Step 10 requires about n^3 flops to compute X, since Y is upper triangular.

Therefore, the proposed algorithm requires approximately $\frac{19n^3}{3}$ flops.

3 The SVD-based Method

We would like to compare the efficiency and accuracy of our proposed Observer-Hessenberg-Triangular algorithm with the SVD-based algorithm proposed earlier in [**?**]. For this purpose, we reproduce the SVD algorithm briefly in the following. Let the matrices F, G and X be partitioned as:

$$F = \begin{bmatrix} F_{1,1} & F_{1,2} & & & \\ & F_{2,2} & F_{2,3} & & \\ & & \cdots & \cdots & \cdots \\ & & & F_{q-1,q-1} & F_{q-1,q} \\ & & & & F_{q,q} \end{bmatrix}, \; X = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_q \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ G_q \end{bmatrix}. \tag{18}$$

where diagonal blocks $F_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, q$ of F, are chosen following the same strategy described in Section **??**.

Using (**??**) in (**??**) and equating the corresponding blocks on each side gives

$$X_i A - F_{ii} X_i E = L_i = \begin{cases} G_1 C, & i = 1 \\ F_{ii-1} X_{i-1} E, & i = 2, \ldots, q \end{cases} \tag{19}$$

For each $i = 1, 2, \ldots, q$, block $\tilde{F}_{ii}$ is assigned with a prescribed spectrum and Y_i is computed to satisfy the generalized Sylvester equation:

$$Y_i A - \tilde{F}_{i,i} Y_i E = J_i = \begin{cases} C, & i = 1 \\ X_{i-1} E, & i = 2, \ldots, q \end{cases}. \tag{20}$$

Let the Singular Value Decomposition (SVD) be given by $Y_i^T = U_i \Sigma_i V_i^T$. Then equation (**??**) becomes

$$V_i \Sigma_i U_i^T A - \tilde{F}_{i,i} V_i \Sigma_i U_i^T E = J_i.$$

and therefore

$$U_i^T A - (\Sigma_i^{-1} V_i^T \tilde{F}_{i,i} V_i \Sigma_i) U_i^T E = \Sigma_i^{-1} V_i^T J_i. \tag{21}$$

Thus, a solution of (**??**) can be computed as:

$$X_i = U_i^T \;,\; F_{i,i} = \Sigma_i^{-1} V_i^T \tilde{F}_{i,i} V_i \Sigma_i, i = 1, 2, \ldots, q \tag{22}$$

$$G_1 = \Sigma_1^{-1} V_1^T \;,\; F_{i,i-1} = \Sigma_i^{-1} V_i^T, i = 2, 3, \ldots, q \tag{23}$$

Flop-count of the SVD Algorithm and comparison with Algorithm 2.1: It can be shown that for unstructured dense matrices, the SVD algorithm in [**?**] requires roughly $\frac{52n^3}{3}$ flops, *which is about 3 times more expensive that Algorithm 2.1.*

Figure 1: Comparison of the Efficiency Between the SVD algorithm and Algorithm **??**.1.

Requirements on the poles of the observer system: in order to guarantee existence of the solution of (**??**), a requirement $\Omega(A, E) \bigcap \Omega(F)$ must be set. This means that the spectra of the matrix F (the desired poles of the observer system) must be set away from the generalized eigenvalues of (A, E). Similar requirement does not exist for the proposed method.

3.1 A comparison of the CPU Time between the SVD algorithm and Algorithm ??.1

A comparison of the efficiency between the SVD algorithm and Algorithm **??**.1 was made using a benchmark example from [**?**]. and the results are reported in Figure **??**. The normalized CPU-time of both the algorithms were compared. This time was calculated by dividing the actual CPU-time by the CPU-time to a call to the the BLAS routine for multiplying two matrices of similar sizes. Benchmarks were done in a SPARC Sun 6 environment under Gnu-Fortran 77, using LAPACK [**?**] and SLICOT [**?**]. *As is seen from Figure **??**, Algorithm **??**.1 requires less CPU-time than the SVD algorithm.*

4 Estimating the State of a Vibrating System

Algorithm **??**.1 can readily be applied to estimate the state and velocity vectors of a second-order control system of the form:

$$\begin{aligned} &M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t) \\ &y(t) = C_1 q(t) + C_2 \dot{q}(t) \end{aligned} \tag{24}$$

where $M, D, K \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and C_1, $C_2 \in \mathbb{R}^{r \times n}$.

It is well-known [**?**] that vibrating and structural dynamics systems, such as bridges, highways, air and space crafts, among others, can be modeled as above. In general, acceleration vector $\ddot{q}(t)$ is easy to estimate. However, the velocity and displacement vectors $\dot{q}(t)$ and $q(t)$ are not easily measurable, though their knowledge is essential to implement a feedback control law of the form:

$$u(t) = v(t) + K_1 q(t) + K_2 \dot{q}(t), \tag{25}$$

Here $K_1, K_2 \in \mathbb{R}^{m \times n}$ are, respectively, the displacement and velocity feedback matrices. This control law must be executed to meet certain design specifications for the system, such as maintaining the stability, controlling dangerous vibrations like resonance, among others (see [**?**]). *Note that the matrices K_1 and K_2 can be computed without the knowledge of the state vectors.*

The system (**??**) can be cast into a descriptor system of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_0 u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{26}$$

where $x(t) = \begin{bmatrix} \dot{q}(t) & q(t) \end{bmatrix}^T$. This can be done in several ways. One of them that preserves the symmetry when the original second-order system is symmetric can be written as [**?**]:

$$E = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix}, \; A = \begin{bmatrix} M & \\ & -K \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 \\ B \end{bmatrix}, \; C = \begin{bmatrix} C_2 & C_1 \end{bmatrix}. \tag{27}$$

In order to estimate the states $x(t)$ of the system (**??**), the descriptor Sylvester-observer equation (**??**) is solved with the matrices E, A, B_0, and C defined by (**??**), using Algorithm **??**.1. After solving (**??**) for matrices X, F, and G, approximations $\hat{q}(t)$ and $\hat{d}(t)$ of $q(t)$ and $\dot{q}(t)$ can be found by solving, in the reduced-order case $(s = 2n - r)$, the system:

$$\begin{bmatrix} XE \\ C \end{bmatrix} \begin{bmatrix} \hat{d}(t) \\ \hat{q}(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ y(t) \end{bmatrix}, \tag{28}$$

where $z(t) \in \mathbb{R}^s$ is the state of the *observer system* given by

$$\dot{z}(t) = Fz(t) + Gy(t) + XB_0 u(t) \tag{29}$$

satisfying any initial condition $z(0) = z_0$.

For a related topic on functional observers for singular systems, see [**?**].

5 Numerical Experiments

5.1 State Estimation of a Satellite System

This model is taken from [**?**]. A communication satellite of mass m orbits around the earth with altitude specified by spherical coordinates $r(t)$, $\theta(t)$, and $\phi(t)$. The orbit is controlled by three orthogonal thrusts: $u_r(t)$, $u_\theta(t)$, and $u_\phi(t)$. The dynamics of this model, given by the Newton's Second Law, is

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 \cos^2\phi + mr\dot{\phi}^2 - k/r^2 + u_r \\ mr^2\ddot{\theta} &= -2mr\dot{r}\dot{\theta} + 2mr^2\dot{\theta}\dot{\phi}\sin\phi/\cos\phi + ru_\theta/\cos\phi \\ mr\ddot{\phi} &= -mr\dot{\theta}^2\cos\phi\sin\phi - 2m\dot{r}\dot{\phi} + u_\phi. \end{aligned} \tag{30}$$

A stationary solution, corresponding to the desired circular equatorial orbit, is given by

$$\begin{bmatrix} r_0(t) & \theta_0(t) & \phi_0(t) \end{bmatrix}^T = \begin{bmatrix} r_0 & \omega_0 t & \omega_0 \end{bmatrix}^T \tag{31}$$

where r_0, ω_0, and k are constants such that $r_0\omega_0^2 = k/\omega_0^2$. Once in orbit, the satellite will remain there if no disturbances happen. However, if it deviates from that orbit, the thrusts are applied to push it back to the right place. Let

$$q(t) = \begin{bmatrix} r(t) - r_0 & \theta(t) - \theta_0 & \phi(t) - \phi_0 \end{bmatrix}^T \tag{32}$$

denote the deviation from the orbit. If the perturbation is very small, then equations (**??**) can be linearized, and a control system (**??**) with three inputs and two outputs can be derived, where

$$M = diag\begin{pmatrix} m & mr_0^2 & mr_0 \end{pmatrix},\ B = diag\begin{pmatrix} 1 & r_0 & 1 \end{pmatrix}$$

$$K = diag\begin{pmatrix} -3m\omega_0^2 & 0 & mr_0\omega_0^2 \end{pmatrix},\ D = \begin{bmatrix} & -2mr_0\omega_0 & \\ 2mr_0\omega_0 & & \\ & & 0 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_r(t) & u_\theta(t) & u_\phi(t) \end{bmatrix}^T.$$

Taking the parameters of this model to be $m = 375.5$ kg, $r_0 = 13400$ m, and $\omega_0 = 0.2618$ rad/sec, the corresponding system matrices in the descriptor form (**??**), in the form given by (**??**), are

$$E = 10^5 \times \begin{bmatrix} 0. & 0. & 0. & 0.0038 & 0. & 0. \\ 0. & 0. & 0. & 0. & 674247.8 & 0. \\ 0. & 0. & 0. & 0. & 0. & 50.317 \\ 0.0038 & 0. & 0. & 0. & -26.346 & 0. \\ 0. & 674247.8 & 0. & 26.346 & 0. & 0. \\ 0. & 0. & 50.317 & 0. & 0. & 0. \end{bmatrix}$$

$$A = 10^5 \times diag\begin{pmatrix} 0.0038 & 674247.8 & 50.317 & 0.0008 & 0.0 & -3.4487 \end{pmatrix}$$

$$C = \begin{bmatrix} 0.2500 & 0.1000 & 0.2500 & 0.4000 & 0.1000 & 0.2000 \\ 0.1000 & 0.2000 & 0.1000 & 0. & 0.3000 & 0. \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 13400. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. \end{bmatrix}^T.$$

The choice of matrices B_0 and C make this system both controllable and observable [**?**]. It can be shown that the associated quadratic pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ has eigenvalues $\pm 0.$, $\pm 0.2618i$ and $\pm 0.2618i$. Therefore, this system is not asymptotically stable.

In order to stabilize this system, we apply the state feedback control law (**??**), with

$$K_1 = 10^1 \begin{bmatrix} -100.5304 & -118169.1827 & 0. \\ -1.4195 & -906602.3462 & 0. \\ 0. & 0. & -77328.6662 \end{bmatrix}$$

$$K_2 = 10^2 \begin{bmatrix} -12.1183 & -30058.1247 & 0. \\ 1.9895 & -139516.5952 & 0. \\ 0. & 0. & -50317. \end{bmatrix}.$$

With the matrices K_1 and K_2 as above, the eigenvalues of the closed-loop system are: $-1/3, -2/3, -1, -4/3, -5/3$ and -2.

We computed impulse responses of the system in three cases: (i) the open-loop system ; (ii) the closed-loop system with actual states; (iii) the closed-loop system with estimated states.

Solving the observer equation (**??**) gives

$$X = \begin{bmatrix} 0.5184 & 0. & 0. & -0.8551 & 0. & -0.0001 \\ 0.8551 & 0. & 0. & 0.5184 & 0. & 0. \\ -0.8975 & 0. & 0. & -0.4410 & 0. & 0. \\ 0.4410 & 0. & 0. & -0.8975 & 0. & -0.0001 \end{bmatrix}$$

$$F = \begin{bmatrix} -7.0237 & -0.1198 & -1.0194 & 6.5465 \\ -0.1929 & -7.9763 & -8.3075 & -1.2937 \\ 0. & 0. & -5.9851 & 0.0292 \\ 0. & 0. & 0.5009 & -5.0149 \end{bmatrix}$$

$$G = \begin{bmatrix} 0. & 0. \\ 0. & 0. \\ -5026.3685 & -617.4799 \\ 2555.7662 & -20804.2752 \end{bmatrix}, \ \Omega(F) = \{ \ -5 \quad -6 \quad -7 \quad -8 \ \}.$$

Figure **??** shows the norm of the impulse responses of the system in the three cases defined above. The initial condition for the observer is $z(0) = 0$. *The plot shows that the control using estimated states is slower than the one using the actual states, but it is clearly successful in bringing the satellite back to the desired orbit.*

Figure 2: Comparison of the Norms of the Impulse Responses for Experiment 5.1

5.2 State Estimation of a System with Singular Stiffness and Damp

Consider a system described by (**??**) with

$$M = \begin{bmatrix} 2. & 1. & 0. \\ 1. & 2. & 0. \\ 0. & 0. & 2. \end{bmatrix}, \ D = \begin{bmatrix} 0. & -0.1 & 0. \\ 0.1 & 0. & 0.2 \\ 0. & -0.2 & 0. \end{bmatrix}, \ K = \begin{bmatrix} 2. & 3. & 0. \\ 3. & 6. & 3. \\ 0. & 3. & 6. \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.3162 & 0. & 0. \\ 0. & 0.3162 & 0. \end{bmatrix}, \ C_2 = \begin{bmatrix} 0.1 & 0. & 0. \\ 0. & 0.1 & 0. \end{bmatrix}$$

$$B = \begin{bmatrix} -3.4641 & 0. \\ 0. & -3.4641 \\ 0. & 0. \end{bmatrix}$$

where $M = M^T > 0, K = K^T$ but $D^T = -D$. Matrices D and K are singular, and system is not asymptotically stable. Furthermore, it can be shown that the associated pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ has eigenvalues $\lambda = \pm 0, \lambda = \pm 2.1469i, \lambda = \pm 1.3193i$ (rounded to 4 figures). It is aimed to stabilize this system with

$$K_1 = \begin{bmatrix} 1.4142 & 0. & 0. \\ 0. & 2.4495 & 0. \end{bmatrix}, \ K_2 = \begin{bmatrix} 0.7071 & 0. & 0. \\ 0. & 1.2247 & 0. \end{bmatrix}$$

so that the eigenvalues of the closed loop system are $\lambda = -1.5765 \pm 2.3476i, \lambda = -0.5795 \pm 1.8520i, -0.0747 \pm 1.6342i$ (rounded to 4 figures).

Reduction to (**??**) through

$$E = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, A = \begin{bmatrix} -D & -K \\ K & 0 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} C_2 & C_1 \end{bmatrix}$$

gives

$$A = \begin{bmatrix} 0. & 0.1000 & 0. & -2. & -3. & 0. \\ -0.1000 & 0. & -0.2000 & -3. & -6. & -3. \\ 0. & 0.2000 & 0. & 0. & -3. & -6. \\ 2. & 3. & 0. & 0. & 0. & 0. \\ 3. & 6. & 3. & 0. & 0. & 0. \\ 0. & 3. & 6. & 0. & 0. & 0. \end{bmatrix}$$
$$E = \begin{bmatrix} 2. & 1. & 0. & 0. & 0. & 0. \\ 1. & 2. & 0. & 0. & 0. & 0. \\ 0. & 0. & 2. & 0. & 0. & 0. \\ 0. & 0. & 0. & 2. & 3. & 0. \\ 0. & 0. & 0. & 3. & 6. & 3. \\ 0. & 0. & 0. & 0. & 3. & 6. \end{bmatrix}, \tilde{B} = \begin{bmatrix} -3.4641 & 0. \\ 0. & -3.4641 \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \end{bmatrix}$$
$$C = \begin{bmatrix} 0.1000 & 0. & 0. & 0.3162 & 0. & 0. \\ 0. & 0.1000 & 0. & 0. & 0.3162 & 0. \end{bmatrix}$$

where $E^T = E$ but $A^T = -A$. Now both E and A are singular.

Solving the observer equation (**??**) through Algorithm **??**.1 gives

$$X = \begin{bmatrix} -1.4161 & -0.5653 & 0.2798 & 0.3223 & 0.3755 & -0.2159 \\ -0.9309 & -1.9657 & 0.9486 & 0.7432 & 0.8336 & -0.5624 \\ 0.3779 & 0.6176 & -0.7346 & 1.7717 & 1.9248 & -1.4655 \\ -0.1453 & -0.6904 & 0.0312 & 0.4259 & 0.5735 & -0.1307 \end{bmatrix}$$
$$F = \begin{bmatrix} -0.1000 & 0. & 0. & 0. \\ -0.1801 & -0.2000 & 0. & 0. \\ 0. & -1.8884 & -0.3000 & 0. \\ 0. & 0. & -0.7482 & -0.4000 \end{bmatrix}$$
$$G = \begin{bmatrix} 14.8788 & 22.3182 \\ 28.0657 & 42.0985 \\ 24.4010 & 36.6015 \\ 32.7670 & 49.1505 \end{bmatrix}$$

where we have chosen the eigenvalues of matrix F (and therefore those of the observer system) to be in the set $\{-0.1, -0.2, -0.3, -0.4\}$.

Figure **??** shows the norm of the impulse responses of the system in the three cases defined in the first experiment. The initial condition for the observer is $z(0) = 0$. *The plot shows that the control using estimated states is slower than the one using the actual states, but it is clearly successful.*

Figure 3: Comparison of the Norms of the Impulse Responses for Experiment 5.2

6 Summary and Conclusion

Implementation of a state-feedback control law in a control system requires the knowledge of the states variables. However, in a practical situation, only a part of the states is measurable and others must be estimated. A well-known approach for state-estimation is via solution of a Sylvester-observer equation and a widely-known algorithm for solution of this equation is the Hessenberg-observer algorithm due to Van Dooren. Although several generalizations of this algorithm and also specialized algorithms for large-scale solution and high performance computing of this equation now exist, no block-wise solution for the generalized Sylvester-observer equation (GSOBE), that arises in the state estimation of descriptor control systems, have been proposed until recently. In 2001, a SVD-based algorithm was developed for this problem by Carvalho and Datta. In this paper, a new algorithm, based on the generalized observer-Hessenberg form, due to Varga, is proposed for GSOBE. The algorithm clearly generalizes the Observer-Hessenberg algorithm for the Sylvester-observer equation and is much more efficient than the SVD algorithm. Some of the distinguished features of this new algorithm are that its implementation and numerical performance rely heavily in Basic Linear Algebra Subroutines (BLAS) Level-3 computations. The last feature makes it very suitable for high-performance computing. There are only a very few high-performance and large-scale algorithms available for control problems and indeed, control theory lags behind other areas of science and engineering in this respect. Thus this new algorithm should be of instant welcome. A natural application of the algorithm is made to state and velocity estimations in second-order models of vibrating structures. Development of an estimation algorithm that works directly with a second-order model without requiring transformation of the model to a state-space form, is currently being investigated.

References

[1] B. Datta. *Numerical Methods for Control Systems Design and Analysis*, Elsevier Academic Press, 2003.

[2] B. Datta and D. Sarkissian. Partial eigenvalue assignment in linear systems : Existence, Uniqueness, and Numerical Methods. *Proc. Mathematical Theory of Networks and Systems* 2002.

[3] S.P. Bhattacharyya and E. deSouza, Pole assignment via Sylvester equation. *System and Control Letters* 1982; **1**: 261-263.

[4] R. K. Cevin III and S. P. Bhattacharyya. Robust and well-conditioned eigenstructures via Sylvester's equation. *Optim. Control and Applied Methods* 1983; **4**: 205-212.

[5] L. H. Keel, J. A. Fleming and S. P. Bhattacharyya. Minimum norm pole assignment via Sylvester's equation. *Contemporary Math.* 1984; **47**: 265-272.

[6] A. Varga. Robust Pole Assignment for descriptor systems. *Proc. MTNS'2000 Symposium* 2000; Perpignan, France.

[7] G. Golub, S. Nash and C. Van Loan. A Hessenberg-Schur method for the problem $AX + XB = C$. *IEEE Trans. Aut. Contr.,* 1979; **AC-24**: 909-913.

[8] P. Van Dooren. Reduced order observers: A new algorithm and proof. *System and Control Letters* 1984; **4**: 243-251.

[9] J. Carvalho and B. Datta. A New block algorithm for the Sylvester- observer equation arising in state-estimation. *Proc. IEEE International CDC* 2001; 3398-3403.

[10] B. Datta and D. Sarkissian. Block algorithms for state estimation and functional observers. *Proc. IEEE International Symposium on Computer-Aided Control Systems Design* 2000; 19-23.

[11] J.Carvalho, K.Datta and Y.Hong. A new block algorithm for full-rank solution of the Sylvester-observer equation. *IEEE Transactions on Automatic Control* 2003; **48 (12)**: 2223 - 2228.

[12] C. Bischof, B. Datta and A. Purkayastha. A parallel algorithm for the Sylvester observer equation. *SIAM J. Sci. Comp.* 1996; **17**: 686-698.

[13] D. Calvetti, B. Lewis and L. Reichel. On the solution of large Sylvester-observer equation. *Num. Lin. Alg. Appl.* 2001; **8** (6-7): 435-452.

[14] B. Datta and Y. Saad. Arnoldi methods for large Sylvester-like observer matrix equations, and an associated algorithm for partial spectrum assignment. *Lin. Alg. Appl.* 1991; **154/156**: 225-244.

[15] L. Dai. *Singular Control Systems*, Springer-Verlag: Berlin: 1989.

[16] M. Fahmy and J. O'Reilly. Observers for descriptor systems. *Int. J. Contr.* 1989; **49**(6): 2013-2028.

[17] T. Kaczorek. Perfect functional observers of singular continuous-time linear systems. *Proc. 5th Portuguese Conference on Automatic Control* 2002; Aveiro, Pt.

[18] J. Carvalho and B. Datta. An Algorithm for Generalized Sylvester-observer Equation in State Estimation of Descriptor Systems. *Proc. IEEE International CDC* 2002.

[19] G. Golub, C. Van Loan. *Matrix Computations*, 3rd. ed., Johns Hopkins Univ. Press: London, 1996.

[20] A. Varga. Computation of Irreducible Generalized State-Space Realizations. *Kybernetika* 1990; **26**: 89-106.

[21] P. Benner, V. Mehrmann, V. Sima, S. VanHuffel and A. Varga. SLICOT-A Subroutine Library in Systems and Control Theory. *Applied and Comput. Control, Signal and Circuits* 1999; **1**:499-539.

[22] D. Luenberger. Observing the state of a linear system. *IEEE Trans. Mil. Electr* 1964; **8**: 74-80.

[23] E.deSouza and S.P.Bhattacharyya. Controllability, observability, and solution of $AX - XB = C$. *Lin. Alg. Appl.* 1981; **39**:167-188.

[24] N. Higham. The Test Matrix Toolbox for Matlab (version 3.0).*Numerical Analysis Report* 276, Sept 95, Manchester Centre for Comput. Math., Univ. Manchester, UK.

[25] J. Dongarra, I. Duff, D. Sorensen and H. Van der Vorst. *Numerical Linear Algebra for High-performance Computers*. SIAM Press, 1998.

[26] D. Inman. *Vibration with Control, Measurement and Stability*. Prentice-Hall: London, 1989.

[27] B. Datta. *Numerical Linear Algebra and Applications*. Brooks/Cole: Pacific Grove, 1995 (second edition is to be published by the Society for Industrial and Applied Mathematics (SIAM), 2008).

[28] J. Carvalho. *State Estimation and Finite Element Model Updating for Vibrating Systems*. Ph.D. Dissertation, Northern Illinois Univ., Dec 2002.

[29] C-T. Chen. *Linear System Theory and Design*, 3rd ed, Oxford Univ. Press: New York, 1999.

[30] A. Laub and F. Arnold. Controllability and observability criteria for multivariable linear second-order models. *IEEE Trans. Aut. Contr.* 1984; **AC-29**(2): 163-165.