ABSTRACT

Name: Daniil R. Sarkissian

Department: Mathematical Sciences

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ABSTRACT

This dissertation is devoted to the study of the two feedback control problems in the matrix second-order and distributed-parameter systems: the partial eigenvalue assignment problem and the partial eigenstructure assignment problem. Contributions are made to both the theory and computations of these problems. The existence and uniqueness results for both the problems in the matrix second-order case and for the partial eigenvalue assignment problem in the distributed-parameter case are derived. New results on orthogonality relations between the eigenvectors (eigenfunctions) of the quadratic matrix (operator) pencil are proved. Computational contributions include development of a novel "direct and partial modal" approach for the solution of these problems. The approach is *direct* because each problem is solved in its own formulation. That is, the problem given in a matrix second-order setting is solved without reformulation to a first-order form. Similarly, the problem when formulated in its own natural distributed-parameter setting is solved without discretization to a reduced-order second-order model. The approach is *partial modal* in the sense that it requires only partial knowledge of eigenvalues and eigenvectors (eigenfunctions). The latter makes the approach completely viable for practical applications because the state-of-the-art techniques are capable of computing only a small part of the spectra of the associated quadratic pencil. This "partial modal" aspect of our solutions is especially remarkable for the distributed-parameter systems since in this case the solution of an infinite dimensional operator problem is obtained by solving a small finite dimensional linear algebraic system. The results on numerical experiments on some real-life examples are given.

NORTHERN ILLINOIS UNIVERSITY

THEORY AND COMPUTATIONS OF PARTIAL EIGENVALUE AND EIGENSTRUCTURE ASSIGNMENT PROBLEMS IN MATRIX SECOND-ORDER AND DISTRIBUTED-PARAMETER SYSTEMS

A DISSERTATION SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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BY
DANIIL R. SARKISSIAN
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Certification:	In accordance with departmental and Graduate School policies, this dissertation is accepted in partial fulfillment of degree requirements.

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Dissertation Director

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CHAPTER 1

INTRODUCTION

The natural model for the vibrating systems, arising in a wide range of applications, especially in the design and analysis of vibrating structures such as bridges, highways, buildings, airplanes, etc., is a homogenous distributed-parameter system of the form

$$\mathbf{M}(x)\frac{\partial^2 \nu(t,x)}{\partial t^2} + \mathbf{C}(x)\frac{\partial \nu(t,x)}{\partial t} + \mathbf{K}(x)\nu(t,x) \quad = \quad 0\,, \tag{1.0.1}$$

where $\mathbf{M}, \mathbf{C} = \mathbf{D} + \mathbf{G}$ and $\mathbf{K}$ are differential operators in the x-domain (spatial domain) of the displacement function $\nu(t,x)$, where for all t the $\nu(t,x)$ belong to some Hilbert space $\mathbb{H}$ that accounts for the boundary conditions of (1.0.1). The operators $\mathbf{M}$, $\mathbf{K}$, $\mathbf{D}$ and $\mathbf{G}$ are, respectively, called *mass*, *stiffness*, *damping*, and *gyroscopic* operators. In many practical applications, $\mathbf{M}$ is self-adjoint and positive definite, $\mathbf{D}$ is self-adjoint, and $\mathbf{G}$ is skew-symmetric. That is, for any nonzero functions $\phi(x), \psi(x) \in \mathbb{H}$ and with the scalar product $(\cdot,\cdot)$, associated with the space $\mathbb{H}$, we have

$$(\mathbf{M}\phi,\phi) > 0, (\mathbf{M}\phi,\psi) = (\phi,\mathbf{M}\psi), (\mathbf{D}\phi,\psi) = (\phi,\mathbf{D}\psi) \text{ and } (\mathbf{G}\phi,\psi) = -(\phi,\mathbf{G}\psi).$$

Though it is desirable to solve a vibration problem in its own natural distributed-parameter settings, very often in practice, due to lack of effective numerical methods to handle a distributed-parameter problem directly, the system (1.0.1) is discretized to a finite-dimensional matrix second-order system of the form:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0, \tag{1.0.2}$$

where $M, C = D + G, K \in \mathbb{R}^{n\times n}$ and $\dot{x}(t)$ and $\ddot{x}(t)$, respectively, denote the first and second derivatives of the time-dependent vector $x(t)$.

In vibration applications, the matrices M, K, D and G are often sparse. They are known, respectively, as *mass*, *stiffness*, *damping*, and *gyroscopic* matrices. In many practical applications, $M = M^T > 0$ (that is, M is symmetric positive definite), $D = D^T$ and $G = G^T$.

Upon separation of variables [1], the system (1.0.2) gives rise to the quadratic eigenvalue problem for the pencil

$$P(\lambda) = \lambda^2 M + \lambda C + K. \tag{1.0.3}$$

The pencil (1.0.3) has $2n$ eigenvalues which are the roots of the equation

$$\det(P(\lambda)) = 0$$

and $2n$ corresponding eigenvectors.

The eigenvalues of $P(\lambda)$ are related to the *natural frequencies* of the homogeneous system (1.0.2), and the eigenvectors are referred to as the *modes* or *mode shapes* of vibration of the system (see [1], [2]).

Dangerous oscillations (called *resonance*) occur when one or more eigenvalues of the pencil (1.0.3) become equal or close to the frequency of the external force. One way to avoid such unwanted oscillations of the vibratory system modelled by (1.0.2) is to apply a control force of the type $f(t) = Bu(t)$, where $B \in \mathbb{R}^{n\times m}$ and $u(t) \in \mathbb{R}^m$. This gives rise to the matrix second-order control system

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) \quad = \quad Bu(t), \tag{1.0.4}$$

which is symbolically written as a pair $(P(\lambda), B)$.

Assuming that the displacement vector $x(t)$ and velocity vector $\dot{x}(t)$ are available, one can choose

$$u(t) \quad = \quad F_1\dot{x}(t) + F_2x(t), \tag{1.0.5}$$

where $F_1, F_2 \in \mathbb{R}^{m\times m}$ are constant matrices. Then the system (1.0.4) becomes the closed-loop system

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) \quad = \quad B(F_1\dot{x}(t) + F_2x(t)), \tag{1.0.6}$$

or, equivalently,

$$M\ddot{x}(t) + (C - BF_1)\dot{x}(t) + (K - BF_2)x(t) \quad = \quad 0. \tag{1.0.7}$$

In vibration control, the matrices B, F_1, and F_2 are known, respectively, as

B – *Control* matrix

F_1 – *Velocity feedback* matrix

F_2 – *State feedback* matrix.

Mathematically, the problem is then to choose the matrices F_1 and F_2 such that the eigenvalues of the associated closed-loop pencil

$$P_c(\lambda) \quad = \quad \lambda^2M + \lambda(C - BF_1) + K - BF_2 \tag{1.0.8}$$

can be altered as required to combat the effects of resonances or to ensure and improve the stability of the system. The problem of finding F_1 and F_2 such that the closed-loop pencil $P_c(\lambda)$ has a desired set of eigenvalues is called the *eigenvalue assignment problem* for the system (1.0.8).

Unfortunately, the known numerical methods [1, 3 – 14] for complete eigenvalue assignment provide satisfactory results only when the ratio n/m is small due to intrinsic worsening of the conditioning of the eigenvalue assignment problem as the dimension of the problem increases. In the multi-input case ($m > 1$), the solution is not unique. In this case, optimization techniques may be employed to select a particular solution that makes the closed-loop eigenvalues as well conditioned as possible.

In a realistic situation, however, only a few eigenvalues are "troublesome," so it makes more sense to alter only those "troublesome" eigenvalues while keeping the rest of the spectrum invariant. This leads to the following variation of the eigenvalue assignment problem:

Problem 1.1 (Partial Eigenvalue Assignment Problem for the Quadratic Matrix Pencil).

Given

1. *Real* $n \times n$ *matrices* $M = M^T > 0$, C, K.

2. *Real* $n \times m$ *(*$m < n$*) control matrix* B.

3. *The self-conjugate subset* $\{\lambda_1, \ldots, \lambda_p\}$, $p < n$ *of the set of open-loop eigenvalues* $\{\lambda_1, \ldots, \lambda_{2n}\}$ *of the pencil (1.0.3) and the corresponding left eigenvector set* $\{y_1, \ldots, y_p\}$.

4. *The self-conjugate set* $\{\mu_1, \ldots, \mu_p\}$ *of scalars.*

Find

Real feedback matrices F_1 and F_2 of order $m \times n$ such that the spectrum of the closed-loop pencil (1.0.8) is the set $S = \{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

While Problem 1.1 is important in its own right it, is to be noted that if the system response needs to be altered by feedback, both eigenvalue assignment as well as eigenvector assignment should be considered. This is because the eigenvalues determine the rate at which system response decays or grows while the eigenvectors determine the shape of the response. Such a problem is called the *eigenstructure assignment problem.* Unfortunately, the eigenstructure assignment problem, in general, is not solvable if the matrix B is given a priori (see [15]). This consideration leads to the following more tractable (but practical) eigenstructure assignment problem:

Problem 1.2 (Partial Eigenstructure Assignment Problem for the Quadratic Matrix Pencil).

Given

1. *Real $n \times n$ matrices $M = M^T > 0$, C, K.*

2. *The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$, $p < n$ of the set of the open-loop eigenvalues $\{\lambda_1, \ldots, \lambda_{2n}\}$ of the pencil (1.0.3) and the corresponding left eigenvector set $\{y_1, \ldots, y_p\}$.*

3. *The self-conjugate sets of scalars $\{\mu_1, \ldots, \mu_p\}$ and the set of vectors $\{x_{c1}, \ldots, x_{cp}\}$, such that $\mu_j = \overline{\mu_k}$ implies $x_{cj} = \overline{x_{ck}}$.*

Find

Real control matrix B of order $n \times m$ $(m < n)$ and real feedback matrices F_1 and F_2 of order $m \times n$ such that the spectrum of the closed-loop pencil (1.0.8) is *the set* $S = \{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ *with* $\{x_{c1}, \ldots, x_{cp}; x_{p+1}, \ldots, x_{2n}\}$ *as the associated eigenvector set, where* $x_{p+1}, \ldots, x_{2n}$ *are the eigenvectors of (1.0.3) corresponding to* $\lambda_{p+1}, \ldots, \lambda_{2n}$.

As in the case of matrix second-order systems, the distributed-parameter systems 1.0.1, upon separation of variables, give rise to the eigenvalue and eigenstructure assignment problems for the quadratic operator pencil

$$\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\,,$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are differential operators.

The concepts of *semi-simple* eigenvalues and *two-fold complete* systems of eigenfunctions that appear in the following statements are defined in Section 3.4. The operator analog of Problem 1.1 can be stated as:

Problem 1.3 (Partial Eigenvalue Assignment Problem for the Quadratic Operator Pencil).

Given

1. *Nonsingular operator* $\mathbf{M}$ *and operators* $\mathbf{C}$ *and* $\mathbf{K}$ *such that the quadratic operator pencil*

$$\mathbf{P}(\lambda) \;=\; \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} \tag{1.0.9}$$

   has a discrete spectrum without finite accumulation points, every eigenvalue of (1.0.9) is semi-simple and the system of eigenfunctions of (1.0.9) is two-fold complete.

2. *The* m *real control functions* $\mathbf{b}_1, \ldots, \mathbf{b}_m$.

3. *The self-conjugate subset* $\{\lambda_1, \ldots, \lambda_p\}$ *of the set of eigenvalues* $\{\lambda_1, \lambda_2, \ldots\}$ *and the corresponding left eigenfunction set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *of (1.0.9).*

4. *The self-conjugate set* $\{\mu_1, \ldots, \mu_p\}$ *of scalars.*

Find

Real feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ *and* $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ *such that the spectrum of the closed-loop pencil*

$$\mathbf{P}_c(\lambda)\phi = \lambda^2 \mathbf{M}\phi + \lambda \left(\mathbf{C}\phi - \sum_{k=1}^{m} (\mathbf{f}_{1k}, \phi)\mathbf{b}_k \right) + \left(\mathbf{K}\phi - \sum_{k=1}^{m} (\mathbf{f}_{2k}, \phi)\mathbf{b}_k \right) \quad (1.0.10)$$

is the set $S = \{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \ldots\}$.

The operator analog of Problem 1.2 can be similarly defined. We, however, will not consider this problem is this dissertation.

Theoretical Contributions:

- The results on existence and uniqueness of solution of all the three problems are derived. The existence and uniqueness results for Problem 1.1 and Problem 1.2 are derived by first deriving these results for the corresponding problems for the first-order system. *These results for the first-order system are also new.*

- The recent results [16 – 21] on the partial eigenvalue and eigenstructure assignment problems for the quadratic matrix pencil and those on the partial eigenvalue assignment for the quadratic operator pencil are derived as specific cases of our results on existence and uniqueness.

- New orthogonality results between the eigenvectors of the quadratic matrix pencil and those between the eigenfunctions of the quadratic operator pencil are proved. Several known results [16, 18] are recovered as special cases.These results, besides their roles in algorithmic solutions of the problems, are of independent interest and are important theoretical results in linear algebra and operator theory applications.

Computational Contributions

- A novel approach is developed for solution of Problems 1.1, 1.2, and 1.3. The approach is "*direct and partial modal*." It is *direct* because each problem is solved in its own given form. That is, Problem 1.1 and Problem 1.2 are solved in matrix second-order settings without reformulation to a first-order form. By doing so, a possibly ill-conditioned matrix inversion is avoided and the exploitable structures, such as symmetry, positive definiteness, sparsity, bandedness, etc., often inherited in the finite-element models are fully exploited in computations. Similarly, Problem 1.3 is solved in its own natural distributed-parameter formulation without discretization to a second-order model. The obtained solution is, therefore, a solution of the problem of the original infinite dimensional model and not the solution of the approximated matrix second-order model. As mentioned in the beginning, this is how the problem should have been solved anyway.

  The proposed approach is *partial modal* in the sense that it requires only partial knowledge of eigenvalues and eigenvectors of the associated quadratic pencil. This makes the approach completely viable for practical applications because the state-of-the-art techniques are capable of computing only a small part of the spectra of quadratic matrix pencils, especially if the problems are large and sparse, which is often the case in design of the large structures, power systems, computer networks, etc. This "partial modal" aspect of our solutions is especially remarkable for Problem 1.3 because this infinite dimensional problem is now solved by solving a small finite dimensional algebraic linear system.

1.1 Dissertation Outline

The outline of the dissertation is as follows:

Chapter 2 is on modelling. Examples are given of how some real-life vibrating problems are modelled using distributed-parameter systems and then it is shown how these distributed-parameter systems are discretized using matrix second-order systems.

The quadratic matrix eigenvalue problem and quadratic operator eigenvalue problem are discussed in Chapter 3. Some of the latest developments on these two problems, including the new orthogonality relations between the eigenvectors of a matrix pencil (eigenfunctions of an operator pencil) are described.

The existing methods for eigenvalue and eigenstructure assignment problem for both quadratic matrix and operator pencils are reviewed in Chapter 4 and their engineering and computational difficulties are mentioned.

Chapters 5 and 6 contain the proposed methods for solving partial eigenvalue and eigenstructure assignment problems for quadratic matrix and operator pencils, respectively.

The results of some numerical experiments on our proposed methods are given in Chapter 7.

Chapter 8 is the final chapter of this dissertation and contains some concluding remarks and future research problems.

1.2 Notation

The following notations are used throughout the dissertation:

n number of states in the second-order model or quadratic matrix pencil $P(\lambda)$ (note that the corresponding first-order system $(A, \hat{B})$ has $2n$ states)

M nonsingular $n \times n$ *mass* matrix of the second-order model (symmetric or undamped gyroscopic model means that M is symmetric and positive definite)

D symmetric $n \times n$ *damping* matrix of the second-order model

G skew-symmetric *gyroscopic* $n \times n$ matrix of the second-order model

C $C = D + G$ (symmetric model means $G = 0$ and undamped gyroscopic model means $D = 0$)

K $n \times n$ *stiffness* matrix of the second-order model (symmetric or undamped gyroscopic model means that K is symmetric)

$x(t)$ $n \times 1$ vector of the state variables in the second-order model satisfying the equation $M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0$

m number of inputs which are used to control the system (single-input case means that $m = 1$ and multi-input case means that $m > 1$)

B $n \times m$ *control* matrix of the second-order model

$h(t)$ $m \times 1$ vector of the control input variables in the controlled second-order model satisfying the equation $M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = Bh(t)$

F_1, F_2 $m \times n$ *velocity* and *position* feedback matrices in the closed-loop second-order model (note that feedback law is $h(t) = F_1\dot{x}(t) + F_2x(t)$)

$P(\lambda)$ open-loop quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$

$P_c(\lambda)$ closed-loop quadratic pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$

λ_j an eigenvalue of the quadratic eigenvalue problem $\det(P(\lambda_j)) = 0$, $j = 1, 2, \ldots, 2n$

x_j the right eigenvector of the quadratic eigenvalue problem for $P(\lambda)$ corresponding to the eigenvalue λ_j and satisfying $P(\lambda_j)x_j = \lambda_j^2 Mx_j + \lambda_j Cx_j + Kx_j = 0$

y_j the left eigenvector of the quadratic eigenvalue problem for $P(\lambda)$ corresponding to the eigenvalue λ_j and satisfying $y_j^H P(\lambda_j) = \lambda_j^2 y_j^H M + \lambda_j y_j^H C + y_j^H K = 0$

Λ the $2n \times 2n$ diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{2n})$

X the $n \times 2n$ matrix of the right eigenvectors $X = (x_1, \ldots, x_{2n})$ satisfying the equation $P(\Lambda)X = MX\Lambda^2 + CX\Lambda + KX = 0$

Y the $n \times 2n$ matrix of the left eigenvectors $Y = (y_1, \ldots, y_{2n})$ satisfying the equation $Y^H P(\Lambda) = \Lambda^2 Y^H M + \Lambda Y^H C + Y^H K = 0$

p the p first eigenvalues $\lambda_1, \ldots, \lambda_p$, where $p < n$, that need to be reassigned to solve the partial eigenvalue or eigenstructure assignment problem

Λ_1, Λ_2 the matrix Λ partitioned as $\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$, where $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$ and $\Lambda_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$

X_1, X_2 the matrix X partitioned as $X = (X_1, X_2)$, where $X_1 = (x_1, \ldots, x_p)$ and $X_2 = (x_{p+1}, \ldots, x_{2n})$

Y_1, Y_2 the matrix Y partitioned as $Y = (Y_1, Y_2)$, where $Y_1 = (y_1, \ldots, y_p)$ and $Y_2 = (y_{p+1}, \ldots, y_{2n})$

μ_j a scalar which replaces the eigenvalue λ_j, $j = 1, 2, \ldots, p$

S the set of closed-loop eigenvalues $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$

Λ_c the $2n \times 2n$ diagonal matrix $\Lambda_c = \text{diag}(\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n})$ containing the eigenvalues of the closed-loop quadratic pencil $P_c(\lambda)$

$\Lambda_{c1}, \Lambda_{c2}$ the matrix Λ_c partitioned as $\Lambda_c = \text{diag}(\Lambda_{c1}, \Lambda_{c2})$, where $\Lambda_{c1} = \text{diag}(\mu_1, \ldots, \mu_p)$ and $\Lambda_{c2} = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$

X_c the $n \times 2n$ matrix of the right eigenvectors $X_c = (x_{c1}, \ldots, x_{c2n})$ satisfying the equation $P(\Lambda_c)X_c = MX_c{\Lambda_c}^2 + (C - BF_1)X_c\Lambda + (K - BF_2)X_c = 0$

X_{c1}, X_{c2} the matrix X_c partitioned as $X_c = (X_{c1}, X_{c2})$, where $Xc_1 = (x_{c1}, \ldots, x_{cp})$ and $X_{c2} = X_2$

Y_c the $n \times 2n$ matrix of the left eigenvectors $Y_c = (y_{c1}, \ldots, y_{c2n})$ satisfying the equation ${Y_c}^H P(\Lambda_c) = {\Lambda_c}^2 {Y_c}^H M + \Lambda_c {Y_c}^H (C - BF_1) + {Y_c}^H (K - BF_2) = 0$

Y_{c1}, Y_{c2} the matrix Y_c partitioned as $Y_c = (Y_{c1}, Y_{c2})$, where $Yc_1 = (y_{c1}, \ldots, y_{cp})$ and $Yc_2 = (y_{cp+1}, \ldots, y_{c2n})$

A the $2n \times 2n$ open-loop state matrix $\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$ of the first-order reformulation of the quadratic pencil $P(\lambda)$

$\lambda(A)$ the spectrum of the matrix A

$\hat{B}$ the $2n \times m$ control matrix $\begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix}$ of the first-order reformulation of the quadratic pencil $P(\lambda)$

$\hat{F}$ the $m \times 2n$ feedback matrix of the first-order reformulation. Note that $\hat{F} = (F_2, F_1)$.

A_c the $2n \times 2n$ closed-loop state matrix $A_c = A - \hat{B}\hat{F}$ of the first-order reformulation of the closed-loop quadratic pencil $P_c(\lambda)$

$\hat{X}$ the $2n \times 2n$ matrix $\hat{X} = (\hat{x}_1, \ldots, \hat{x}_{2n})$ of the right eigenvectors of A. Note that $\hat{x}_j = \begin{pmatrix} x_j \\ \lambda_j x_j \end{pmatrix}$, $j = 1, \ldots, 2n$, and $\hat{X} = \begin{pmatrix} X \\ X\Lambda \end{pmatrix}$.

$\hat{Y}$ the $2n \times 2n$ matrix $\hat{Y} = (\hat{y}_1, \ldots, \hat{y}_{2n})$ of the left eigenvectors of A. Note that $\hat{y}_j = \begin{pmatrix} \lambda_j M^H y_j + C^H y_j \\ M^H y_j \end{pmatrix}$, $j = 1, \ldots, 2n$, and $\hat{Y}^H = \big(\Lambda Y^H M + Y^H C,\, Y^H M\big)$.

$\hat{X}_c$ the $2n \times 2n$ matrix $\hat{X}_c = (\hat{x}_{c1}, \ldots, \hat{x}_{c2n})$ of the right eigenvectors of A_c.

$\hat{Y}_c$ the $2n \times 2n$ matrix $\hat{Y}_c = (\hat{y}_{c1}, \ldots, \hat{y}_{c2n})$ of the left eigenvectors of A_c.

For the operator pencil, the following additional notations are used:

M nonsingular *mass* operator of the distributed-parameter model (symmetric or undamped gyroscopic model means that **M** is self-adjoint and positive definite)

D self-adjoint *damping* operator of the distributed-parameter model

G skew-symmetric *gyroscopic* operator of the distributed-parameter model

C $\mathbf{C} = \mathbf{D} + \mathbf{G}$ (symmetric model means $\mathbf{G} = 0$ and undamped gyroscopic model means $\mathbf{D} = 0$)

K *stiffness* operator of the distributed-parameter model (symmetric or undamped gyroscopic model means that **K** is self-adjoint)

t, x time and spatial variables, respectively, in the distributed-parameter model

$\nu(t,x)$ displacement function in the distributed-parameter model

m number of inputs which are used to control the system (single-input case means that $m = 1$ and multi-input case means that $m > 1$)

$\mathbf{b}_k$ *control* function of the distributed-parameter model, $1 \leq k \leq m$

$\mathbf{f}_{1k}, \mathbf{f}_{2k}$ *velocity* and *position* feedback functions in the closed-loop distributed-parameter model

$\mathbf{P}(\lambda)$ open-loop quadratic operator pencil

$\mathbf{P}_c(\lambda)$ closed-loop quadratic operator pencil

$\mathbf{w}_j, \mathbf{v}_j$ the right and left eigenfunctions, respectively, of the quadratic eigenvalue problem for the operator pencil $\mathbf{P}(\lambda)$ corresponding to the eigenvalue λ_j

$\mathbf{w}_{cj}, \mathbf{v}_{cj}$ the right and left eigenfunction, respectively, of the closed-loop quadratic eigenvalue problem for the operator pencil $\mathbf{P}_c(\lambda)$ corresponding to the eigenvalue μ_j if $j \leq p$ or to the eigenvalue λ_j if $j > p$

CHAPTER 2

MODELLING

In this chapter, we illustrate how vibration of some well known physical systems can be modelled with distributed mass, stiffness, and damping parameters. Such systems are called *distributed-parameter systems*. We give two examples, one on small oscillations of a travelling string and the other on small oscillations of a rotating flexible shaft. We show that the first one is an *undamped gyroscopic system* and the second one is a *damped gyroscopic system*.

In practice a distributed-parameter system is often discretized to a matrix second-order system. We show in some detail how to do this using finite element techniques. Both the distributed-parameter and the discretized second-order systems will be used to illustrate our proposed numerical methods for feedback control problems.

2.1 Small Oscillations of a Travelling String

Consider the small oscillations of a uniform string travelling with constant velocity γ over two fixed supports at $x = 0$ and $x = L$. This example is both simple enough to be used to illustrate the proposed method throughout the dissertation and general enough to compute the essence of such problems as stabilization of towed sonar arrays or dampening the waves created by the high-speed trains. The motion of the moving string, shown in Figure 2.1, is governed by the partial differential equation

$$\nu_{tt} + 2\gamma\nu_{xt} + (\gamma^2 - c^2)\nu_{xx} = 0, \tag{2.1.1}$$

where $0 < x < L$, $t > 0$, $\gamma^2 < c^2$, with boundary conditions given by

$$\nu(0,t) = \nu(L,t) = 0, \tag{2.1.2}$$

see *e.g.* [22].

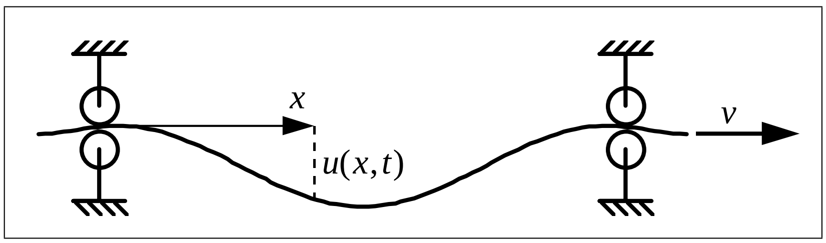

Figure 2.1: Small Oscillations of Travelling String.

Let us now define the operators **M**, **G**, and **K** by

$$\mathbf{M}v = v\,,\ \mathbf{G}v = 2\gamma\frac{\partial v}{\partial x}\,,\ \mathbf{K}v = (\gamma^2 - c^2)\frac{\partial^2 v}{\partial x^2}. \tag{2.1.3}$$

Then, with respect to the *scalar product* (also called *inner product* or *sesquilinear form*)

$$(v, w) = \int_0^L \overline{v(x)}w(x)\ dx\,,$$

we have

$$\begin{aligned}(\mathbf{M}v, w) &= \overline{\int_0^L \overline{w(x)}v(x)\ dx} = \overline{(\mathbf{M}w, v)} = (v, \mathbf{M}w)\,,\\ (\mathbf{M}v, v) &= \int_0^L |v(x)|^2\ dx \geq 0\,.\end{aligned}$$

Integrating by parts yields

$$(\mathbf{K}v, w) = -\overline{\int_0^L \overline{(\gamma^2 - c^2)w''(x)}v(x)\ dx} = (v, \mathbf{K}w)\,,$$

and

$$(\mathbf{K}v, v) \quad = \quad \int_0^L (\gamma^2 - c^2)|v'(x)|^2\ dx > 0\,,$$

in view of the boundary conditions (2.1.2). Since $v'(x)$ does not vanish identically, $\mathbf{K}$ is a self-adjoint positive definite operator. Another integration gives

$$(\mathbf{G}v, w) = -\overline{\int_0^L \overline{2\gamma w'(x)}v(x)\ dx} = -(v, \mathbf{G}w)\,.$$

Thus the system defined by (2.1.1) and (2.1.2) is an operator system with self-adjoint operators $\mathbf{M}$ and $\mathbf{K}$ and the skew-symmetric operator $\mathbf{G}$. Such a system is called an *undamped gyroscopic operator system.*

We now show how to discretize this operator system to a matrix second-order system. Let $\{\phi_k(x)\}_{k=1}^{\infty}$ be a complete system of functions and let each $\phi_k(x)$ satisfy the essential boundary conditions (2.1.2), for example, $\phi_k(x) = \sin\left(\dfrac{k\pi x}{L}\right)$. Then there exists coefficients $\{v_k(t)\}_{k=1}^{\infty}$ such that

$$\nu(x, t) = \sum_{k=1}^{\infty} v_k(t)\phi_k(x)\,.$$

Now requiring that

$$(\phi_j, M\frac{\partial^2 \nu_n}{\partial t^2}) + (\phi_j, G\frac{\partial \nu_n}{\partial t}) + (\phi_j, K\nu_n) = 0,\ \text{ for } j = 1, 2, \ldots, n, \tag{2.1.4}$$

where

$$\nu_n(x,t) = \sum_{k=1}^{n} v_k(t)\phi_k(x), \tag{2.1.5}$$

we obtain the system of n equations with n unknowns $v_1, \ldots, v_n$. Integrating $(\phi_j, K\nu_n)$ by parts and writing (2.1.4) in the form (1.0.2) we obtain

$$M\ddot{v}(t) + G\dot{v}(t) + Kv(t) = 0, \text{ where } v(t) = (v_1(t), v_2(t), \ldots, v_n(t))^T,$$

where the jk^{th} elements of M, G, and K are, respectively, given by

$$(\phi_j, \phi_k)\,,\ 2\gamma(\phi_j, \frac{\partial \phi_k}{\partial x})\,,\ \text{and } (c^2 - \gamma^2)(\frac{\partial \phi_j}{\partial x}, \frac{\partial \phi_k}{\partial x})\,. \tag{2.1.6}$$

2.2 Small Oscillations of the Damped Rotating Flexible Shaft

Consider an elastic shaft rotating with a constant angular velocity ω_z and loaded with a constant axial force η. Let x denote the axial coordinate and $\nu_j(t,x)$ the transverse displacements in the principal directions $(j = 1, 2)$. Let EI_j be the flexural rigidities in the principal directions, L be the length, and $\rho(x)$ be the density of the shaft.

Suppose that the internal damping is given by $\rho(x)d_i(\dot{\nu}_1, \dot{\nu}_2)$, which is related to transverse velocities in the rotating frame, and the external damping is given by $\rho(x)d_e(\dot{\nu}_1 - \omega_z\nu_2, \dot{\nu}_2 - \omega_z\nu_1)$, which is related to the velocities relative to stationary coordinates.

Hence, the oscillations of the damped rotating flexible shaft are governed by the system of partial differential equations

$$\rho\frac{\partial^2\nu_1}{\partial t^2}+(d_i+d_e)\rho\frac{\partial\nu_1}{\partial t}-2\omega_z\rho\frac{\partial\nu_2}{\partial t}+EI_1\frac{\partial^4\nu_1}{\partial x^4}+\eta\frac{\partial^2\nu_1}{\partial x^2}-\omega_z^2\rho\nu_1-d_e\rho\omega_z\nu_2=0$$
$$\rho\frac{\partial^2\nu_2}{\partial t^2}+(d_i+d_e)\rho\frac{\partial\nu_2}{\partial t}+2\omega_z\rho\frac{\partial\nu_1}{\partial t}+EI_2\frac{\partial^4\nu_2}{\partial x^4}+\eta\frac{\partial^2\nu_2}{\partial x^2}-\omega_z^2\rho\nu_2+d_e\rho\omega_z\nu_1=0.$$

Let's assume that the ends of the shaft are either simply supported or clamped so that the boundary conditions at each end are either

$$\nu_j=0,\ \frac{\partial^2\nu_j}{\partial x^2}=0\quad\text{or}\quad\nu_j=0,\ \frac{\partial\nu_j}{\partial x}=0,\qquad\text{for } j=1,2. \tag{2.2.7}$$

As in Section 2.1, it can be shown that with respect to the scalar product

$$(v,w)=\sum_{j=1}^{2}\int_0^L\overline{v_j(x))}w_j(x)\ dx,\ \text{where } v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}\ \text{and } w=\begin{pmatrix}w_1\\ w_2\end{pmatrix} \tag{2.2.8}$$

the above equations could be written in the form (1.0.1) as

$$\mathbf{M}\nu_{tt}+\mathbf{C}\nu_t+\mathbf{K}\nu=0,\ \text{where } \nu=\begin{pmatrix}\nu_1(x,t)\\ \nu_2(x,t)\end{pmatrix}, \tag{2.2.9}$$

with

$$\mathbf{M}=\begin{pmatrix}\rho & 0\\ 0 & \rho\end{pmatrix}, \tag{2.2.10}$$

$$\mathbf{C}=\mathbf{D}+\mathbf{G}=\begin{pmatrix}(d_i+d_e)\rho & 0\\ 0 & (d_i+d_e)\rho\end{pmatrix}+\begin{pmatrix}0 & -2\omega_z\rho\\ 2\omega_z\rho & 0\end{pmatrix}, \tag{2.2.11}$$

and

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1 = \tag{2.2.12}$$
$$\begin{pmatrix} EI_1\frac{\partial^4}{\partial x^4} + \eta\frac{\partial^2}{\partial x^2} - \omega_z^2\rho & 0 \\ 0 & EI_2\frac{\partial^4}{\partial x^4} + \eta\frac{\partial^2}{\partial x^2} - \omega_z^2\rho \end{pmatrix} + \begin{pmatrix} 0 & -d_e\rho\omega_z \\ d_e\rho\omega_z & 0 \end{pmatrix}.$$

The above model is a *damped gyroscopic model*.

This model (2.2.9) could be discretized in the same way as the traveling string model in the Section 2.1. Thus, taking the system of functions

$$\psi_k(x) = \begin{pmatrix} \phi_k(x) \\ 0 \end{pmatrix} \text{ and } \psi_{n+k}(x) = \begin{pmatrix} 0 \\ \phi_k(x) \end{pmatrix} \text{ for } k = 1, 2, \ldots, n$$

and approximating the true solution $\nu(x,t)$ as

$$\nu_{2n}(x,t) = \sum_{k=1}^{2n} \upsilon_k(t)\psi_k(x)\,,$$

we obtain an $2n \times 2n$ system of matrix second-order ordinary differential equations, just like the system (2.1.6) was obtained from finite element representation (2.1.5).

Let M_n, E, U_1, and U_2 be the matrices where jk^{th} elements are, respectively, defined by

$$(\rho\phi_j, \phi_k)\,,\ (\eta\frac{\partial\phi_j}{\partial x}, \frac{\partial\phi_k}{\partial x})\,,\ (EI_1\frac{\partial^2\phi_j}{\partial x^2}, \frac{\partial^2\phi_k}{\partial x^2})\,,\ \text{and } (EI_2\frac{\partial^2\phi_j}{\partial x^2}, \frac{\partial^2\phi_k}{\partial x^2})\,.$$

Then

$$M = \begin{pmatrix} M_n & 0 \\ 0 & M_n \end{pmatrix},$$

$$C = (d_i + d_e)\begin{pmatrix} M_n & 0 \\ 0 & M_n \end{pmatrix} + 2\omega_z \begin{pmatrix} 0 & -M_n \\ M_n & 0 \end{pmatrix},$$

and

$$K = \begin{pmatrix} U_1 - E - \omega_z^2 M_n & 0 \\ 0 & U_2 - E - \omega_z^2 M_n \end{pmatrix} + d_e\omega_z \begin{pmatrix} 0 & -M_n \\ M_n & 0 \end{pmatrix}.$$

2.2.1 Damped Symmetric Model

When ω_z is close enough to zero, the model (2.2.9) can be simplified by neglecting the gyroscopic terms (that is, by setting $\omega_z = 0$) to obtain the following self-conjugated system of partial differential equations governing the oscillations of the damped flexible shaft:

$$\begin{aligned} \rho\frac{\partial^2\nu_1}{\partial t^2} + (d_i + d_e)\rho\frac{\partial\nu_1}{\partial t} + EI_1\frac{\partial^4\nu_1}{\partial x^4} + \eta\frac{\partial^2\nu_1}{\partial x^2} &= 0 \\ \rho\frac{\partial^2\nu_2}{\partial t^2} + (d_i + d_e)\rho\frac{\partial\nu_2}{\partial t} + EI_2\frac{\partial^4\nu_2}{\partial x^4} + \eta\frac{\partial^2\nu_2}{\partial x^2} &= 0. \end{aligned}$$

When the same scalar product (2.2.8) is used, the above equations could be written as

$$\mathbf{M}\nu_{tt} + \mathbf{D}\nu_t + \mathbf{K}_0\nu = 0, \text{ where } \nu = \begin{pmatrix} \nu_1(x,t) \\ \nu_2(x,t) \end{pmatrix}. \quad (2.2.13)$$

Since $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}_0$ are all self-adjoint operators, we call (2.2.13) a symmetric model.

The same finite-element technique can be applied to this simplified symmetric model to discretize it. In general, symmetric models are often useful to describe the vibration of the body when its rotation is small enough to be considered negligible. It is well known (see proof of Corollary 3.1 below) that the left eigenvectors of the symmetric model (2.2.13) are the conjugates of its right eigenvectors, which simplifies the solution procedures for Problems 1.1 and 1.2 for such systems.

2.2.2 Undamped Gyroscopic Model

When the damping coefficients d_i and d_e are close enough to zero, the model (2.2.9) can be simplified by neglecting the damping terms to obtain the following gyroscopic system of partial differential equations governing the oscillations of the rotating flexible shaft:

$$\rho\frac{\partial^2\nu_1}{\partial t^2} - 2\omega_z\rho\frac{\partial\nu_2}{\partial t} + EI_1\frac{\partial^4\nu_1}{\partial x^4} + \eta\frac{\partial^2\nu_1}{\partial x^2} - \omega_z^2\rho\nu_1 = 0$$
$$\rho\frac{\partial^2\nu_2}{\partial t^2} + 2\omega_z\rho\frac{\partial\nu_1}{\partial t} + EI_2\frac{\partial^4\nu_2}{\partial x^4} + \eta\frac{\partial^2\nu_2}{\partial x^2} - \omega_z^2\rho\nu_2 = 0.$$

When the same scalar product (2.2.8) is used, the above equations could be written as

$$\mathbf{M}\nu_{tt} + \mathbf{G}\nu_t + \mathbf{K}_0\nu = 0, \text{ where } \nu = \begin{pmatrix} \nu_1(x,t) \\ \nu_2(x,t) \end{pmatrix}. \qquad (2.2.14)$$

Since both $\mathbf{M}$ and $\mathbf{K}_0$ are symmetric operators, and $\mathbf{G}$ is skew-symmetric, we call (2.2.14) a gyroscopic model.

The same finite-element technique could be used for this simplified symmetric model to discretize the model. In general, gyroscopic models arise in an important particular case when the damping forces are small compared to the gyroscopic forces in a vibratory system and could be neglected. It is well known (see proof of Corollary 3.2 in the next chapter) that the set of left eigenvectors of the gyroscopic system (2.2.14) is a permutation of a set of its right eigenvectors, which leads to the simplified solution procedure for Problems 1.1 and 1.2 for such systems.

CHAPTER 3

THEORY AND COMPUTATIONS OF QUADRATIC MATRIX AND OPERATOR EIGENVALUE PROBLEMS

In this chapter we study the matrix quadratic eigenvalue problem arising from the matrix second-order model (1.0.2) and the operator quadratic eigenvalue problem arising in the distributed-parameter model (1.0.1).

3.1 Quadratic Matrix Eigenvalue Problem

In this section we define the matrix quadratic eigenvalue problem. Let us start with the following preliminary definition:

Definition 3.1 *A matrix function* $P : \mathbb{C} \to \mathbb{C}^{n\times n}$ *is called the* quadratic matrix pencil *if* $P(\lambda) = \lambda^2 M + \lambda C + K$*, where* $M \neq 0$ *and* C *and* K *are constant* $n \times n$ *matrices.*

"The rather strange use of the word *pencil* comes from optics and geometry. An aggregate of (light) rays converging to a point does suggest the sharp end of a pencil and, by a natural extension, the term came to be used for any one-parameter family of curves, spaces, matrices, or other mathematical objects" [23, p. 302].

The pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ is very often referred to as a *lambda matrix*, *matrix polynomial*, or *matrix bundle* of degree 2 in the mathematical literature [24, 25].

Definition 3.2 *A scalar* $\lambda \in \mathbb{C}$ *such that* $\det(P(\lambda)) = 0$ *is called an* eigenvalue *of the quadratic pencil* P*. The set of all eigenvalues is called the* spectrum *of* P*.*

Definition 3.3 *The non zero vectors* x *and* y *are, respectively, called the* right *and* left eigenvectors*, corresponding to the eigenvalue* λ *of the quadratic pencil* $P(\lambda) = \lambda^2 M + \lambda C + K$ *if*

$$\left(\lambda^2 M + \lambda C + K\right) x = 0 \tag{3.1.1}$$

and

$$y^H \left(\lambda^2 M + \lambda C + K\right) = 0, \tag{3.1.2}$$

where y^H *is the conjugate transpose of the vector* y*.*

Definition 3.4 *The triplet* (λ, x, y) *is called the eigenpair of* P*.*

Definition 3.5 *The pairs* (λ, x) *and* (λ, y) *are called, respectively, right and left eigenpairs of* P*.*

The *quadratic eigenvalue problem* is the problem of determining all the eigenvalues and the corresponding eigenvectors of the given quadratic pencil $P(\lambda)$. Note that the standard eigenvalue problem

$$Ax = \lambda x$$

and the generalized eigenvalue problem

$$Ax = \lambda Bx$$

are special cases of this problem (see (3.1.3) and (3.1.12)).

Definition 3.6 *The pencil P is called* singular *if for any $\lambda \in \mathbb{C}$ the matrix $P(\lambda)$ is singular. Otherwise the pencil is called* regular*.*

In this dissertation we restrict ourselves to regular quadratic pencils.

3.1.1 Two Standard Approaches for the Quadratic Eigenvalue Problem

There are two standard approaches for solving the quadratic eigenvalue problem: first, via the standard eigenvalue problem; second, via the generalized eigenvalue problem.

Approach 1: Computing the Eigenvalues and Eigenvectors of P via the Standard Eigenvalue Problem.

This approach is based on the following result:

Theorem 3.1 (Relation Between the Quadratic and the Standard Eigenvalue Problems).
A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ with the corresponding right eigenvector x and the left eigenvector y if and only if λ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \qquad (3.1.3)$$

with the corresponding right eigenvector $\hat{x}$ and the left eigenvector $\hat{y}$ such that

$$\hat{x} = \begin{pmatrix} x \\ \lambda x \end{pmatrix} \text{ and } \hat{y} = \begin{pmatrix} \lambda M^H y + C^H y \\ M^H y \end{pmatrix}. \qquad (3.1.4)$$

Proof. Since (λ, x, y) is an eigenpair of the quadratic pencil P, we must have

$$\left(\lambda^2 M + \lambda C + K\right) x = 0 \quad \text{and} \quad y^H \left(\lambda^2 M + \lambda C + K\right) = 0. \tag{3.1.5}$$

From (3.1.5), it then follows that

$$A\hat{x} \;=\; \begin{pmatrix} \lambda x \\ -M^{-1}(K + \lambda C)x \end{pmatrix} = \lambda \begin{pmatrix} x \\ \lambda x \end{pmatrix} = \lambda \hat{x}$$

and

$$\hat{y}^H A \;=\; \left(-y^H K,\, \lambda y^H M + y^H C - y^H C\right) = \left(\lambda^2 y^H M + \lambda y^H C,\, \lambda y^H M\right) = \lambda \hat{y}^H,$$

which proves that $(\lambda, \hat{x}, \hat{y})$ is an eigenpair of the matrix A.

Next, suppose that λ is an eigenvalue of A and $\hat{x}$ is the associated right eigenvector. Then

$$A\hat{x} = \lambda \hat{x}, \quad \text{where} \quad \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}. \tag{3.1.6}$$

The equation (3.1.6) can be written as

$$\hat{x}_2 \;=\; \lambda \hat{x}_1 \tag{3.1.7}$$

and

$$-M^{-1}K\hat{x}_1 - M^{-1}C\hat{x}_2 \;=\; \lambda \hat{x}_2. \tag{3.1.8}$$

Substituting the equation (3.1.7) into (3.1.8) and multiplying by M on the left, we get

$$-K\hat{x}_1 - \lambda C\hat{x}_1 = \lambda^2 M\hat{x}_1.$$

This shows that λ is the eigenvalue of P with the right eigenvector $\hat{x}_1$.

Similarly, if $\hat{y}$ is the right eigenvector of A associated with the eigenvalue λ, then

$$\hat{y}^H A = \lambda \hat{y}^H, \quad \text{where} \quad \hat{y}^H = (\hat{y}_1^H, \hat{y}_2^H). \tag{3.1.9}$$

The equation (3.1.9) can be written as

$$-\hat{y}_2^H M^{-1} K = \lambda \hat{y}_1^H \tag{3.1.10}$$

and

$$\hat{y}_1^H - \lambda \hat{y}_2^H M^{-1} C = \lambda \hat{y}_2^H. \tag{3.1.11}$$

Substituting the equation (3.1.10) into the equation (3.1.11) after multiplication by λ we obtain

$$-(\hat{y}_2^H M^{-1})K - \lambda(\hat{y}_2^H M^{-1}) = \lambda^2(\hat{y}_2^H M^{-1})M,$$

which shows that $(\hat{y}_2^H M^{-1})$ is the left eigenvector.

Theorem 3.1 says that the $2n$ eigenvalues of the pencil $P(\lambda)$ are the eigenvalues of the matrix $A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$. Furthermore, if $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$ and $\hat{y} = (\hat{y}_1^H, \hat{y}_2^H)^H$ are, respectively, the right and left eigenvectors of A corresponding to an eigenvalue λ, then the respective right and left eigenvectors x and y of P are determined by

$$x = \hat{x}_1 \text{ and } y^H = \hat{y}_2^H M^{-1}.$$

Definition 3.7 *The eigenvalue λ of the matrix A is called* semi-simple *if it has the same algebraic and geometric multiplicities, that is, if it has m linearly independent* *eigenvectors, where* m *is multiplicity of the root* λ *of the characteristic polynomial* $\det(A - sI)$ *of the matrix* A.

Definition 3.8 *The eigenvalue* λ *of the quadratic pencil* P *is called* semi-simple *if* λ *is the semi-simple eigenvalue of the matrix* A *corresponding to this quadratic pencil.*

Requiring all eigenvalues to be semi-simple means that there is no need to introduce *Jordan chains* (also called the *associated vectors*) to describe the eigenstructure of our system. Since arbitrarily small perturbations can destroy the Jordan form, and for the reasons of numerical stability, in this dissertation we restrict ourselves to eigenvalue problems with semi-simple eigenvalues.

Approach 2: Computing the Eigenvalues and Eigenvectors of P via the Generalized Eigenvalue Problem.

The next theorem shows that the eigenvalue problem for the pencil $P(\lambda)$ is equivalent to the following generalized eigenvalue problem:

Theorem 3.2 (Relation Between the Quadratic and the Generalized Eigenvalue Problems).
A scalar $\lambda \in \mathbb{C}$ *is an eigenvalue of the quadratic pencil* $P(\lambda) = \lambda^2 M + \lambda C + K$ *with the corresponding right eigenvector* x *and the left eigenvector* y *if and only if* λ *is an eigenvalue of the generalized eigenproblem* $A - \lambda B$, *where*

$$A = \begin{pmatrix} 0 & K \\ K & C \end{pmatrix} \text{ and } B = \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} \tag{3.1.12}$$

with $\hat{x} = \begin{pmatrix} x \\ \lambda x \end{pmatrix}$ *and* $\hat{y} = \begin{pmatrix} y \\ \bar{\lambda} y \end{pmatrix}$ *as right and left eigenvectors, respectively, corresponding to* λ.

Proof. Proof of this theorem is analogous to the proof of Theorem 3.1.

Remark 3.1 *Reduction of the symmetric quadratic pencil $P(\lambda)$ to symmetric generalized eigenvalue problems of other types of linear pencils is also possible (see [2] for details).*

Clearly, the quadratic eigenvalue problem for the pencil $P(\lambda)$ can be solved by solving the standard eigenvalue problem for the matrix A given by (3.1.3) or the generalized eigenvalue problem for the pair (A, B) given by (3.1.12).

Unfortunately, if the standard eigenvalue problem given by Theorem 3.1 is used to compute the eigenvalues and eigenvectors of the quadratic pencil $P(\lambda)$, then the matrix M has to be inverted, and, if it is ill-conditioned, then the eigenvalues and eigenvectors will not be computed accurately. Furthermore, all the exploitable properties such as definiteness, sparsity, bandedness, etc., of the coefficient matrices M, C, and K, usually offered by a practical problem, will be completely destroyed. The use of a generalized eigenvalue problem will double the dimension of the problem and also destroy the definiteness and the bandedness properties of the coefficient matrices, though, in some cases symmetry can be preserved.

Because of the above-mentioned computational difficulties, it is not possible to compute the eigenvalues and eigenvectors of the quadratic pencil $P(\lambda)$ accurately, especially if the dimension n is large, which is often the case in many practical applications. Fortunately, in our proposed solution technique of the partial eigenstructure and eigenvalue assignment problems given in Chapter 5, we need only a small part of the spectrum and the associated eigenvectors. The next section shows how to compute them in an efficient way.

3.2 Computation of the Partial Spectrum for Quadratic Eigenvalue Problem

We briefly review various methods that are commonly used to compute part of the spectrum of the quadratic eigenvalue problem and discuss their advantages and limitations.

3.2.1 Shifted and Inverted Quadratic Eigenvalue Problem

The well known and popular "*shift and invert method*" ([26], see also [2] and [25]) is an iterative technique to compute a small number of eigenvalues and the corresponding eigenvectors of a matrix. For practical applications, it is useful that the structures of the matrices M, C, and K of the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$, such as symmetry, bandedness, sparsity, etc., are exploited in computations. The shift and invert method is capable of doing so. In this section we discuss shift and invert strategy for the pencil $P(\lambda)$.

Using the shift $\lambda = \mu + \sigma$, the eigenvalue problem for the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ can be transformed into the equivalent eigenvalue problem for the pencil $P_1(\mu)$ given by

$$P_1(\mu) \quad = \quad \mu^2 M + \mu(C + 2\sigma M) + (K + \sigma C + \sigma^2 M). \tag{3.2.13}$$

In particular, μ is an eigenvalue of $P_1(\mu)$ if and only if $\mu + \sigma$ is an eigenvalue of $P(\lambda)$, since $P(\mu + \sigma) = P_1(\mu)$.

Similarly, if 0 is not an eigenvalue of $P(\lambda)$, then the inversion $\lambda = 1/\mu$ transforms the eigenvalue problem for $P(\lambda)$ to the eigenvalue problem for the quadratic pencil

$$P_2(\mu) \quad = \quad \mu^2 K + \mu C + M. \tag{3.2.14}$$

Combining the above shift and invert transformations (3.2.13) and (3.2.14), the quadratic eigenvalue problem for $P(\lambda)$ is transformed to the eigenvalue problem for

$$P_\sigma(\mu) \quad = \quad \mu^2(K + \sigma C + \sigma^2 M) + \mu(C + 2\sigma M) + M, \tag{3.2.15}$$

with $\lambda = \sigma + \dfrac{1}{\mu}$ or equivalently $\mu = \dfrac{1}{\lambda - \sigma}$.

By definition, the pencil $P(\lambda)$ is regular if there exists a shift σ such that the matrix $P(\sigma) = K + \sigma C + \sigma^2 M$ is nonsingular.

Theorem 3.3 (Shift and Invert Method for the Quadratic Eigenvalue Problem).

For any scalars $a_1, a_2, \ldots, a_{2n}$ and any scalar σ not in the spectrum of $P(\lambda)$ the sequence of vectors

$$z_j \quad = \quad \sum_{k=1}^{2n} \frac{a_k}{(\lambda_k - \sigma)^j}\, x_k \;\; \textit{for} \;\; j = 0, 1, 2, \ldots \tag{3.2.16}$$

satisfy the recurrence relation

$$(\sigma^2 M + \sigma C + K) z_{j+1} + (2\sigma M + C) z_j + M z_{j-1} \quad = \quad 0\,, \tag{3.2.17}$$

where each (λ_k, x_k) is the (right) eigenpair of the quadratic matrix pencil $P(\lambda)$.

Proof. By Theorem 3.1 the recurrence relation

$$\left(\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} - \sigma \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right) \begin{pmatrix} z_{j+1} \\ \tilde{z}_{j+1} \end{pmatrix} \quad = \quad \begin{pmatrix} z_j \\ \tilde{z}_j \end{pmatrix} \tag{3.2.18}$$

is satisfied for the sequence of vectors

$$\begin{pmatrix} z_j \\ \tilde{z}_j \end{pmatrix} \quad = \quad \sum_{k=1}^{2n} \frac{a_k}{(\lambda_k - \sigma)^j} \begin{pmatrix} x_k \\ \lambda_k x_k \end{pmatrix} \quad \text{for} \;\; j = 0, 1, 2, \ldots,$$

where the (λ_k, x_k)'s are the (right) eigenpairs of both the quadratic matrix pencil $P(\lambda)$ and the standard eigenvalue problem, specified by the Theorem 3.1.

Rewriting the equation (3.2.18) as

$$\tilde{z}_{j+1} = z_j + \sigma z_{j+1} \tag{3.2.19}$$

$$-Kz_{j+1} - C\tilde{z}_{j+1} - \sigma M\tilde{z}_{j+1} = M\tilde{z}_j \tag{3.2.20}$$

and substituting (3.2.19) into (3.2.20), we obtain

$$-Kz_{j+1} - C(z_j + \sigma z_{j+1}) - M(\sigma z_j + \sigma^2 z_{j+1}) = M(z_{j-1} + \sigma z_j)\,. \tag{3.2.21}$$

Collecting similar terms in (3.2.21), we get the desired recurrence relation (3.2.17).

Based on Theorem 3.3 we can state the following algorithm:

Algorithm 3.4 (Shift and Invert Method for the Quadratic Eigenvalue Problem).

Inputs: *The $n \times n$ matrices M, C, and K; scalar σ; and tolerance ε.*

Outputs: *Right eigenvector x of the quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda C + K$.*

Assumptions: *$P(\sigma)$ is non singular.*

Step 1. *Set $x_0 = 0$ and choose an arbitrary vector x_1 or unit norm.*

Step 2. *Iterate steps 3 and 5 for $j = 1, 2, \ldots$ until x_j converges. When convergence occurs, accept the last x_j as x.*

Step 3. *Solve the linear system*

$$(\sigma^2 M + \sigma C + K)\tilde{x}_{j+1} = -(2\sigma M + C)x_j - Mx_{j-1}. \tag{3.2.22}$$

Step 4. *Set* $x_{j+1} = x_{j+1}/||\tilde{x}_{j+1}||_2$ *and* $x_j = x_j/||\tilde{x}_{j+1}||_2$.

Step 2. *Check the convergence:* $\frac{||x_{j+1} - x_j||}{||x_j||} < \varepsilon$.

Just as "the shift-and-invert algorithm" for the standard eigenvalue problem can be accelerated by the use of the Rayleigh quotient instead of some fixed shift σ, the new Rayleigh quotient for the quadratic pencil, which is proposed in the next section, can also be used to accelerate the performance of Algorithm 3.4.

3.2.2 The Rayleigh Quotient for the Quadratic Pencil

As we just saw in the last section, the shift and invert method requires a suitable approximation to the eigenvalue λ for its implementation. A standard way to do this is to use the Rayleigh quotient. In this section, we show how to compute the *Rayleigh quotient for the quadratic pencil.*

For the standard eigenvalue problem $Ax = \lambda x$, the Rayleigh quotient equation is defined by (see, *e.g.* [2]):

$$\lambda = \frac{y^H Ax}{y^H x} = \frac{x^H Ax}{x^H x} = \frac{y^H Ay}{y^H y}\,. \tag{3.2.23}$$

If the matrix A is hermitian and only the right eigenpairs are needed, then the Rayleigh quotient becomes $\lambda = \frac{x^H Ax}{x^H x}$.

Similarly, for the generalized eigenproblem $Ax = \lambda Bx$, the Rayleigh quotient is defined by (see, *e.g.* [7])

$$y^H Ax = \lambda(y^H Bx) \text{ or } \lambda = \frac{y^H Ax}{y^H Bx}\,, \tag{3.2.24}$$

provided that $y^H Bx \neq 0$.

Unfortunately, the straightforward generalization of this concept to find the Rayleigh quotient for the quadratic pencil $P(\lambda)$ given by

$$\lambda^2 x^H M x + \lambda x^H C x + x^H K x = 0\,, \tag{3.2.25}$$

where x is an approximate right eigenvector of the quadratic pencil, leads to the problem of "spurious roots." That is, we know that at least one root of (3.2.25) must be the eigenvalue λ, corresponding to the right eigenvector x. But the second root of the equation (3.2.25) generally has no meaning for the given quadratic eigenproblem. The problem then is to distinguish the "good" root, which accelerates the convergence in Steps 3 and 4 of Algorithm 3.4, from the spurious one, which breaks the convergence. It is customary to take the root with the smallest residual norm $||((\lambda^{(k)})^2 M + \lambda^{(k)} C + K)x^{(k)}||_2$, but that one may not be the right choice for the purpose of accelerating the convergence. For details, see [27, §9.2.1].

To eliminate this difficulty, in this dissertation we propose to approximate a *pair* of eigenvalues λ_i and λ_j, where $1 \leq i \neq j \leq 2n$, simultaneously using the modified Rayleigh quotient equation

$$\lambda^2 y_i^H M x_j + \lambda y_i^H C x_j + y_i^H K x_j = 0\,. \tag{3.2.26}$$

If the exact i^{th} left eigenvector y_i and the exact j^{th} right eigenvector x_j are used, then both roots of (3.2.26) are meaningful. Indeed, multiplying the equation (3.1.2) for the left eigenpair (λ_i, y_i) by x_j on the right, we obtain

$$y_i^H \left(\lambda_i^2 M + \lambda_i C + K\right) x_j = 0$$

and thus the equation (3.2.26) must have solution λ_i. Similarly, multiplying the equation (3.1.1) by y_i^H on the left we prove that λ_j must also be a solution of (3.2.26).

Since a quadratic equation cannot have more than two roots, the equation (3.2.26) has exactly roots λ_i and λ_j, and no spurious roots are possible.

Example 1 (An illustrative example)

Following [27] we consider the quadratic pencil

$$P(\lambda) = \lambda^2 \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 1 \\ 0 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}$$

with four different pairwise conjugate eigenvalues

$$\alpha_{1,2} = -0.9396 \pm 1.5749i \text{ and } \alpha_{3,4} = -0.0049 \pm 0.6296i.$$

The associated right eigenvectors are

$$x_{1,2} = (1, -2.4756 \mp 0.9779i)^T \text{ and } x_{3,4} = (1, 0.0326 \mp 0.0132i)^T$$

and the associated left eigenvectors are

$$y_{1,2} = (1, -4.7175 \pm 0.5233i)^T \text{ and } y_{3,4} = (1, 0.0439 \pm 0.0780i)^T .$$

Using the modified Rayleigh quotient (3.2.26) we recover the eigenvalues α_1 and α_2 from the equation

$$0 = \lambda^2 y_1^H M x_2 + \lambda y_1^H C x_2 + y_1^H K x_2 = (44.09 + 11.84i)\lambda^2 + (82.86 + 22.25i)\lambda + 148.29 + 39.81i$$

and eigenvalues α_3 and α_4 from the equation

$$0 = \lambda^2 y_3^H M x_4 + \lambda y_3^H C x_4 + y_3^H K x_4 = (5.1190 + 0.0595i)\lambda^2 + (0.0499 + 0.0006i)\lambda + 2.0295 + 0.0236i .$$

No spurious roots have contaminated our Rayleigh quotients.

On the other hand, using the equation (3.2.25) with the eigenvector x_1, we obtain the equation

$$0 = \mu^2 x_1^H M x_1 + \mu\, x_1^H C x_1 + x_1^H K x_1 =$$
$$(25.9131 - 0.9779i)\mu^2 + (47.1192 - 0.9779i)\mu + 87.0196\,,$$

which has the following two roots: $\mu_1 = -0.9396 + 1.5749i$ and $\mu_2 = -0.8776 - 1.6057i$. The eigenvalue λ_1 is correctly recovered by μ_1; however, μ_2 is spurious.

3.2.3 The Jacobi-Davidson Method

Originally, the Jacobi-Davidson method was developed to compute the partial spectrum of the standard large and sparse eigenvalue problem

$$Ax = \lambda x. \tag{3.2.27}$$

It is based on the idea of projection. Given an orthogonal basis V of the low-dimensional search subspace, the large eigenvalue problem (3.2.27) is approximated by the "projected" eigenvalue problem

$$V^H A V s = \theta V^H V s. \tag{3.2.28}$$

The system (3.2.28) is a smaller eigenvalue problem and, therefore, can be efficiently solved by any of the standard methods. If (θ, s) is the eigenpair of (3.2.28), then the Ritz value θ is an approximate eigenvalue of (3.2.27) corresponding to the eigenvector approximated by the Ritz vector $x = Vs$.

The method is iterative in nature. At each iteration the basis V is expanded by the approximate solution $t \perp x$, of the correction equation

$$(I - xx^H)(A - \theta I)(I - xx^H)t \;=\; -r, \tag{3.2.29}$$

where $x = Vs$ and $r = Ax - \theta x$. For stability reasons, the basis of the search subspace is constructed to be orthonormal. The new basis vector is the orthogonal complement of t with respect to the previous basis vectors.

It can be shown that if the correction equation (3.2.29) is solved with sufficient accuracy then the asymptotic rate of convergence to the eigenpair of (3.2.27) is at least quadratic. In practice, however, it is often more efficient to approximate the solution to the equation (3.2.29) by a fast iterative numerical method, such as, for example, a small number of GMRES steps [2, 28]. Although this increases the number of Jacobi-Davidson iterations, each iteration becomes considerably less expensive. If the desired eigenvalue is well separated from the other eigenvalues, then the Jacobi-Davidson method converges almost quadratically. In other cases, the convergence is linear and its rate depends on the relative separation of the desired eigenvalue.

The idea can be generalized to compute the partial spectrum of the quadratic pencil $P(\lambda)$. Thus, the large quadratic eigenvalue problem for the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ is projected onto a low-dimensional subspace spanned by the columns of V, which leads to the small eigenvalue problem for the quadratic pencil

$$P_V(\theta) \;=\; \theta^2 V^H M V + \theta V^H C V + V^H K V \tag{3.2.30}$$

that can be solved by any direct method. The best (for example, closest to some target value or with the largest real part) eigenvalue θ of this projected eigenproblem is selected. If s is the right eigenvector corresponding to θ, then the Ritz vector

$x = Vs$ approximates the eigenvector of $P(\lambda)$ corresponding to the approximate eigenvalue θ of $P(\lambda)$.

If the approximation is not satisfactory, then the search subspace spanned by the orthogonal columns of the matrix V has to be expanded by a vector t that is determined by the Jacobi-Davidson correction equation

$$\left(I - \frac{px^H}{x^H p}\right) P(\theta)(I - xx^H)t \quad = \quad -r \tag{3.2.31}$$

where $x = Vs$, $r = P(\theta)x = (\theta^2 M + \theta C + K)x$, and $p = P'_V(\theta)x = (2\theta M + C)x$. Then V is replaced by the result of the modified Gram-Schmidt method applied to the matrix (V, t).

The process is repeated until the desired eigenpair is detected, that is until the residual vector r becomes small enough. If the dimension of the search subspace becomes too large, then the process could be restarted with a new search subspace determined by the few best eigenpairs of the projected low-dimensional problem (3.2.30).

Based on the above discussion we can state the following algorithm:

Algorithm 3.5 (Jacobi-Davidson Method for the Quadratic Eigenvalue Problem).

Inputs: *The $n \times n$ matrices $M, C,$ and K and tolerance ε.*

Outputs: *An eigenvalue θ and right eigenvector x of the quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda C + K$.*

Step 1. *Choose an $n \times m$ orthonormal matrix V, and compute $W_0 = KV$, $W_1 = CV$, $W_2 = MV$ and $M_i = V^H W_i$ for $i = 0, 1, 2$.*

Step 2. *Iterate steps 3 through 9 until convergence.*

Step 3. *Compute the right eigenpairs* (θ_j, s_j) *of the projected pencil*

$$(\theta_j^2 M_2 + \theta_j M_1 + M_0)s_j = 0\,.$$

Step 4. *Select the desired eigenpair* (θ, s) *with* $||s||_2 = 1$.

Step 5. *Compute* $x = Vs$, $r = P(\theta)x$ *and* $p = P'_V(\theta)x$.

Step 6. *If* $||r|| < \varepsilon$ *then STOP.*

Step 7. *Solve (approximately) the following linear system for* $t \perp x$*:*

$$\left(I - \frac{px^H}{x^H p}\right) P(\theta)(I - xx^H)t \;\; = \;\; -r\,.$$

Step 8. *Orthogonalize* t *against* V, $v = t/||t||_2$ *and compute for* $i = 0, 1, 2$*:*

$$w_0 = Kv, w_1 = Cv, w_2 = Mv \text{ and } M_i = \begin{pmatrix} M_i & V^H w_i \\ v^H W_i & v^H w_i \end{pmatrix}.$$

Step 9. *Expand* $V = (V, v)$ *and* $W_i = (W_i, w_i)$, $i = 0, 1, 2$.

In the numerical experiments of this dissertation, the more numerically stable *Jacobi-Davidson-style QZ algorithm* described in [29] and available as MATLAB function JDQZ on the web site of the authors is used. Detailed discussion about Jacobi-Davidson method is outside of the scope of this dissertation (see [27, 29, 30] for more details).

3.3 Orthogonality Relations Between the Eigenvectors

In this section, we first derive **new orthogonality relations** between the eigenvectors of a quadratic matrix pencil. *It is then shown that the recent orthogonality results of the symmetric definite quadratic matrix pencil [16, 18] as well as the well known results on the orthogonality between the eigenvectors of a symmetric matrix and of a symmetric definite linear pencil follow as special cases.*

First, we state and prove a well known result on the orthogonality of the eigenvectors of a matrix [2, 28].

Theorem 3.6 (Orthogonality of the Eigenvectors of a Matrix).
Let $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* $A \in \mathbb{C}^{n\times n}$ *and let* $\hat{X}$ *and* $\hat{Y}$ *be, respectively, the right and left eigenvector matrices. Assume that* $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} = \emptyset$. *Partition* $\hat{X} = (\hat{X}_1, \hat{X}_2)$ *and* $\hat{Y} = (\hat{Y}_1, \hat{Y}_2)$, *where* $\hat{X}_1 = (\hat{x}_1, \ldots, \hat{x}_p)$, $\hat{X}_2 = (\hat{x}_{p+1}, \ldots, \hat{x}_{2n})$, $\hat{Y}_1 = (\hat{y}_1, \ldots, \hat{y}_p)$, *and* $\hat{Y}_2 = (\hat{y}_{p+1}, \ldots, \hat{y}_{2n})$.

Then

$$\hat{Y}_1^H \hat{X}_2 = 0 \tag{3.3.32}$$

and

$$\hat{Y}_1^H A \hat{X}_2 = 0. \tag{3.3.33}$$

If, in addition, A is real symmetric, then

$$\hat{X}_1^T \hat{X}_2 = 0 \quad \textit{and} \quad \hat{X}_1^T A \hat{X}_2 = 0. \tag{3.3.34}$$

Proof. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Since $\hat{X}$ and $\hat{Y}$ are eigenvector matrices, we have

$A\hat{X} = \hat{X}\Lambda$ and $\hat{Y}^H A = \Lambda \hat{Y}^H$. This implies that

$$\Lambda \hat{Y}^H \hat{X} = \hat{Y}^H A \hat{X} = \hat{Y}^H \hat{X} \Lambda. \tag{3.3.35}$$

The equation (3.3.35) can be written as

$$N\Lambda = \Lambda N, \text{ where } N = (n_{ij})_{i,j=1}^n = \hat{Y}^H \hat{X}.$$

From the last equation, we have $n_{ij}\lambda_i = \lambda_j n_{ij}$, which shows that $n_{ij} = 0$ whenever $1 \le i \le p < j \le n$ or $1 \le j \le p < i \le n$. In matrix notation, this implies relation (3.3.32).

The relation (3.3.33) follows immediately from (3.3.32), since

$$\hat{Y}_1^H A \hat{X}_2 = \Lambda_1 \hat{Y}_1^H \hat{X}_2 = 0,$$

where $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$.

Finally, if the matrix A is symmetric then $\hat{Y}^H = \hat{X}^T$, which proves (3.3.34).

We next state and prove an orthogonality relation between the eigenvectors of the linear pencil $A - \lambda B$.

Theorem 3.7 (Orthogonality of the Eigenvectors of a Linear Matrix Pencil).

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the linear pencil $A - \lambda B$ with $A, B \in \mathbb{C}^{n \times n}$ and let X and Y be, respectively, the right and left eigenvector matrices. Assume that $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} = \emptyset$. Partition $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. Then

$$Y_1^H B X_2 = 0 \tag{3.3.36}$$

and

$$Y_1^H A X_2 = 0. \tag{3.3.37}$$

If, in addition, both A and B are real symmetric, then

$$X_1^T B X_2 = 0 \quad \textit{and} \quad X_1^T A X_2 = 0. \tag{3.3.38}$$

Proof. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Since X and Y are eigenvector matrices, we have $AX = BX\Lambda$ and $Y^H A = \Lambda Y^H B$. This implies that

$$\Lambda(Y^H BX) = Y^H AX = (Y^H BX)\Lambda\,.$$

The rest of the proof is similar to that of Theorem 3.6.

The following theorem establishes the orthogonality relations for the quadratic pencil using its connection with the standard eigenvalue problem given in Theorem 3.1 and the orthogonality relations in Theorem 3.6. The relation (3.3.39) turns out to play a key role in our later developments.

Theorem 3.8 (Orthogonality of the Eigenvectors of Quadratic Pencil). *Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of the $n \times n$ quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ and let X and Y be, respectively, the right and left eigenvector matrices. Assume that $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$. Partition $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$.*

Then

$$\Lambda_1 Y_1^H M X_2 \Lambda_2 - Y_1^H K X_2 = 0 \tag{3.3.39}$$

and

$$\Lambda_1 Y_1^H M X_2 + Y_1^H M X_2 \Lambda_2 + Y_1^H C X_2 = 0, \tag{3.3.40}$$

where $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ *with* $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ *and* $\Lambda_2 = \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$.

Proof. By Theorem 3.1, the matrix $A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$ has the right eigenvector matrix $\hat{X}$ and the left eigenvector matrix $\hat{Y}$ given by

$$\hat{X} = \begin{pmatrix} X \\ X\Lambda \end{pmatrix} \text{ and } \hat{Y}^H = \big(\Lambda Y^H M + Y^H C,\, Y^H M\big).$$

From equation (3.3.32) of Theorem 3.6, we then have

$$0 = \hat{Y}_1^H \hat{X}_2 = \big(\Lambda_1 Y_1^H M + Y_1^H C,\, Y_1^H M\big) \begin{pmatrix} X_2 \\ X_2\Lambda_2 \end{pmatrix} =$$
$$\Lambda_1 Y_1^H M X_2 + Y_1^H M X_2 \Lambda_2 + Y_1^H C X_2,$$

proving the relation (3.3.40).

Similarly, from the equation (3.3.33) we obtain (3.3.39) as follows:

$$0 = \hat{Y}_1^H A \hat{X}_2 = \big(\Lambda_1 Y_1^H M + Y_1^H C,\, Y_1^H M\big) \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \begin{pmatrix} X_2 \\ X_2\Lambda_2 \end{pmatrix} =$$
$$\Lambda_1 Y_1^H M X_2 \Lambda_2 + Y_1^H C X_2 \Lambda_2 - Y_1^H M M^{-1} K X_2 - Y_1^H M M^{-1} C X_2 \Lambda_2 =$$
$$\Lambda_1 Y_1^H M X_2 \Lambda_2 - Y_1^H K X_2.$$

We now show that the recent results on the orthogonality relations for the symmetric definite quadratic pencil [16] and those for the gyroscopic pencil [18] can be recovered from the above theorem. The result of Corollary 3.1 was established in [16] and that of Corollary 3.2 was established in [18].

Corollary 3.1 (Orthogonality of the Eigenvectors of Symmetric Definite Quadratic Pencil).

Consider the symmetric definite quadratic pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K, \text{ where } M = M^T > 0, D = D^T, K = K^T. \quad (3.3.41)$$

Let Λ be the eigenvalue matrix and X be the corresponding eigenvector matrix.

Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ are distinct, then the matrix

$$\Lambda X^T M X \Lambda - X^T K X \quad (3.3.42)$$

is a diagonal matrix.

Proof.

The transposed matrix equation for the (right) eigenvectors of symmetric quadratic pencil $P(\lambda)$ is

$$\Lambda^2 X^T M + \Lambda X^T D + X^T K = 0.$$

Since all eigenvalues are distinct, the relation

$$\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$$

is satisfied. Also, for any $1 \leq p \leq 2n$, the equation (3.3.39) becomes

$$\Lambda_1 X_1^T M X_2 \Lambda_2 - X_1^T K X_2 = 0, \text{ (since } Y_1^H = X_1^T), \quad (3.3.43)$$

where $X_1 = (x_1, \ldots, x_p)$ and $X_2 = (x_{p+1}, \ldots, x_{2n})$. Therefore, the matrix $\Lambda X^T M X \Lambda - X^T K X$ is lower triangular, since (3.3.43) implies that for each $1 \leq i \leq p$ and

$p + 1 \le j \le 2n$, the ij^{th} element of this matrix is zero. Because M and K are symmetric, the matrix $\Lambda X^T M X \Lambda - X^T K X$ is upper triangular as well. Thus, it is diagonal.

Corollary 3.2 (Orthogonality of the Eigenvectors of the Undamped Gyroscopic Quadratic Pencil).

Consider the undamped gyroscopic quadratic pencil

$$P(\lambda) = \lambda^2 M + \lambda G + K, \ \textit{where}\ M = M^T > 0, G = -G^T, K = K^T. \tag{3.3.44}$$

Let the eigenvalue matrix Λ *and the (right) eigenvector matrix* X *be partitioned as*

$$\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2) = \operatorname{diag}(\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n})$$

and

$$X = (X_1; X_2) = (x_1, \ldots x_p; x_{p+1}, \ldots, x_{2n}).$$

Assume that $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$ *and* $\{\lambda_1, \ldots, \lambda_p\} = -\overline{\{\lambda_1, \ldots, \lambda_p\}}$, *then*

$$\Lambda_1 X_1^H M X_2 \Lambda_2 + X_1^H K X_2 = 0\,. \tag{3.3.45}$$

Proof. Since $P(\lambda)$ is gyroscopic, $P(\lambda)^H = P(-\overline{\lambda})$. Thus, if λ is an eigenvalue of $P(\lambda)$, so is $-\overline{\lambda}$. In matrix terms, this means that there exists a permutation matrix T such that

$$T = T^T = T^{-1} \quad \text{and} \quad -\overline{\Lambda} = T \Lambda T. \tag{3.3.46}$$

Now, the matrix equation for the right eigenvectors of the undamped gyroscopic quadratic pencil is

$$MX\Lambda^2 + GX\Lambda + KX = 0.$$

Transposing and conjugating this equation we obtain

$$(-\overline{\Lambda})^2 X^H M + (-\overline{\Lambda}) X^H G + X^H K = 0.$$

From (3.3.46), it then follows that $Y^H = TX^H$, where Y is the matrix of left eigenvectors. Moreover, the condition $\{\lambda_1, \ldots, \lambda_p\} = -\overline{\{\lambda_1, \ldots, \lambda_p\}}$ implies that

$$T_1 = T_1^T = T_1^{-1}, \; -\overline{\Lambda_1} = T\Lambda_1 T, \text{ and } T_1 Y_1^H = X_1,$$

where T_1 is the leading $p \times p$ submatrix of T.

From the equations (3.3.39) and (3.3.46), we then obtain that

$$0 = (T_1\Lambda_1 T_1)(T_1 Y_1^H) M X_2 \Lambda_2 - (T_1 Y_1^H) K X_2 = -\Lambda_1 X_1^H M X_2 \Lambda_2 - X_1^H K X_2.$$

3.4 Eigenvalue Problem for Quadratic Operator Pencil

In this section we define the quadratic eigenvalue problem for an operator pencil. Let us start with following preliminary definitions, which we tailor to be similar to the Definitions 3.2 – 3.8 for the quadratic matrix eigenvalue problem.

Definition 3.9 *Let $\mathbb{H}$ be a Hilbert space with an appropriate scalar product*

$$(\alpha\phi(x), \beta\psi(x)) \;=\; \overline{\alpha}\beta(\phi(x), \psi(x)) = \overline{(\beta\psi(x), \alpha\phi(x))}\,. \tag{3.4.47}$$

An operator function $\mathbf{P} : \mathbb{C} \to \mathbb{H}$ *is called a* quadratic operator pencil *if*

$$(\mathbf{P}(\lambda)\phi)(x) = \lambda^2\mathbf{M}(x)\phi(x) + \lambda\mathbf{C}(x)\phi(x) + \mathbf{K}(x)\phi(x)\,, \tag{3.4.48}$$

where $\phi \in \mathbb{H}$*, and* $\mathbf{M}$*,* $\mathbf{C}$*, and* $\mathbf{K}$ *are differential operators from* $\mathbb{H}$ *to* $\mathbb{H}$*.*

The operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ is very often referred to as the *operator polynomial* or *operator bundle* of degree 2 in the mathematical literature [31, 32].

Definition 3.10 *A scalar* $\lambda \in \mathbb{C}$ *such that the operator* $\mathbf{P}(\lambda)$ *is not invertible is called an* eigenvalue *of the quadratic operator pencil* $\mathbf{P}(\lambda)$*. The set of all eigenvalues is called the* spectrum *of* $\mathbf{P}(\lambda)$*.*

Definition 3.11 *The non zero functions* $w(x), v(x) \in \mathbb{H}$ *are, respectively, called the* right *and* left eigenfunctions*, corresponding to the eigenvalue* λ *from the* discrete spectrum *of the quadratic operator pencil* $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ *if for all functions* $\phi(x) \in \mathbb{H}$

$$\lambda^2(\phi, \mathbf{M}w) + \lambda(\phi, \mathbf{C}w) + (\phi, \mathbf{K}w) = 0 \tag{3.4.49}$$

and

$$\lambda^2(v, \mathbf{M}\phi) + \lambda(v, \mathbf{C}\phi) + (v, \mathbf{K}\phi) = 0. \tag{3.4.50}$$

Recall that an operator A^* is an adjoint operator if for all $\phi, \psi \in \mathbb{H}$

$$(A^*\phi, \psi) = (\phi, A\psi)\,.$$

If we rewrite (3.4.49) and (3.4.50) as

$$\mathbf{P}(\lambda)w = (\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})w = 0 \tag{3.4.51}$$

and

$$\mathbf{P}(\lambda)^* v = (\lambda^2\mathbf{M}^* + \lambda\mathbf{C}^* + \mathbf{K}^*)v = 0, \tag{3.4.52}$$

respectively, then Definition 3.11 looks more similar to Definition 3.3 for eigenvectors of the quadratic matrix pencil.

Definition 3.12 *The triplet* (λ, w, v) *is called the eigenpair of the operator pencil* $\mathbf{P}(\lambda)$*.*

Definition 3.13 *The pairs* (λ, w) *and* (λ, v) *are called, respectively, right and left eigenpairs of the pencil* $\mathbf{P}(\lambda)$*.*

The *eigenvalue problem for the quadratic operator pencil* is the problem of determining all the eigenvalues and the corresponding eigenfunctions of the given quadratic operator pencil.

Definition 3.14 *The operator pencil* $\mathbf{P}(\lambda)$ *is called* singular *if for all* $\lambda \in \mathbb{C}$ *the operator* $\mathbf{P}(\lambda)$ *is not invertible. Otherwise the operator pencil is called* regular*.*

In this dissertation we restrict ourselves to regular quadratic operator pencils.

Definition 3.15 *The associated functions* $w_1, \ldots, w_n \in \mathbb{H}$ *of the operator pencil* $\mathbf{P}(\lambda)$ *at the right eigenpair* (λ, w_0) *are defined by the relations*

$$\begin{aligned}
&\mathbf{P}(\lambda)w_1 + \frac{1}{1!}\frac{\partial \mathbf{P}(\lambda)}{\partial\lambda}w_0 = 0\,,\\
&\quad\vdots\\
&\mathbf{P}(\lambda)w_n + \frac{1}{1!}\frac{\partial \mathbf{P}(\lambda)}{\partial\lambda}w_{n-1} + \cdots + \frac{1}{n!}\frac{\partial^n \mathbf{P}(\lambda)}{\partial\lambda^n}w_0 = 0
\end{aligned}$$

Note that such a sequence of functions $w_0, w_1, \ldots, w_n$ is often called the *Jordan chain.*

The important but somewhat technically complicated concept of multiple completeness in the spectral theory of pencils has been described in [33]. To simplify the exposition we use Lemma 13.3 of [32] to define *two-fold completeness* for the quadratic operator pencil $\mathbf{P}(\lambda)$ as follows.

Definition 3.16 *The system of eigenfunctions and associated functions of the quadratic operator pencil* $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ *is* two-fold complete *in* $\mathbb{H}$ *if the system of eigenfunctions and associated functions of the generalized operator eigenvalue problem of the linear pencil*

$$\begin{pmatrix} K & C \\ 0 & I \end{pmatrix} - \lambda \begin{pmatrix} 0 & -M \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & \mathbf{C}+\lambda\mathbf{M} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathbf{P}(\lambda) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\lambda I & I \end{pmatrix}$$

is complete in $\mathbb{H}^2$*, where* $\mathbb{H}^2$ *is the orthogonal sum of two copies of* $\mathbb{H}$*.*

Definition 3.17 *The eigenvalue* λ *of the quadratic operator pencil* $\mathbf{P}(\lambda)$ *is called* semi-simple *if* λ *is not a finite accumulation point of the spectrum and* λ *has no associated functions.*

The detailed spectral theory of the quadratic operator pencil is beyond the scope of this dissertation; therefore, we make the following assumption for the rest of this dissertation:

Assumption 3.9 *The open-loop quadratic operator pencil* $\mathbf{P}(\lambda)$ *has discrete spectrum without finite accumulation points, every eigenvalue of* $\mathbf{P}(\lambda)$ *is semi-simple, and the system of eigenfunctions of* $\mathbf{P}(\lambda)$ *is two-fold complete.*

There is a large body of research that deals with the spectral theory of operators, in particular we mention [24, 25, 31, 32, 34, 35, 36, 37, 38, 39, 40].

3.5 Orthogonality Relations Between the Eigenfunctions of the Quadratic Operator Pencil

In this section, we derive **new orthogonality relations** between the eigenfunctions of a quadratic operator pencil and then *show how a recent known result [18] on the orthogonality between the eigenfunctions of the undamped gyroscopic quadratic operator pencil can be obtained as a special case.*

We first establish an orthogonality relation between the eigenfunctions of a damped quadratic operator pencil. This result (the relation (3.5.54)) plays a key role in our later developments.

Theorem 3.10 (Orthogonality of the Eigenfunctions of Quadratic Operator Pencil).
Let $\lambda_1, \lambda_2, \ldots$ *be the eigenvalues of the damped quadratic operator pencil* $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ *and let* $\mathbf{w}_1, \mathbf{w}_2, \ldots$ *and* $\mathbf{v}_1, \mathbf{v}_2, \ldots$ *be, respectively, the right and left eigenfunctions of* $\mathbf{P}(\lambda)$.

If $\lambda_j \neq \lambda_k$, *then*

$$\lambda_j\lambda_k(\mathbf{v}_j, \mathbf{M}\mathbf{w}_k) - (\mathbf{v}_j, \mathbf{K}\mathbf{w}_k) \quad = \quad 0 \tag{3.5.53}$$

and

$$(\lambda_j + \lambda_k)(\mathbf{v}_j, \mathbf{M}\mathbf{w}_k) + (\mathbf{v}_j, \mathbf{C}\mathbf{w}_k) \quad = \quad 0\,. \tag{3.5.54}$$

Proof. Since $\mathbf{w}_k$ and $\mathbf{v}_j$ are, respectively, the right and left eigenfunctions of $\mathbf{P}(\lambda)$, the equations (3.4.49) and (3.4.50) become, respectively,

$$\lambda_k^2(\phi, \mathbf{M}\mathbf{w}_k) + \lambda_k(\phi, \mathbf{C}\mathbf{w}_k) + (\phi, \mathbf{K}\mathbf{w}_k) \quad = \quad 0 \tag{3.5.55}$$

for all $\phi \in \mathbb{H}$ and

$$\lambda_j^2(\mathbf{v}_j, \mathbf{M}\psi) + \lambda_j(\mathbf{v}_j, \mathbf{C}\psi) + (\mathbf{v}_j, \mathbf{K}\psi) \quad = \quad 0 \tag{3.5.56}$$

for all $\psi \in \mathbb{H}$. Subtracting (3.5.55) with $\phi = \overline{\lambda}_j \mathbf{v}_j$ from (3.5.56) with $\psi = \lambda_k \mathbf{w}_k$, we obtain

$$\lambda_k \lambda_j (\lambda_k - \lambda_j)(\mathbf{v}_j, \mathbf{M}\mathbf{w}_k) + (\lambda_j - \lambda_k)(\mathbf{v}_j, \mathbf{K}\mathbf{w}_k) \quad = \quad 0\,. \tag{3.5.57}$$

Since $\lambda_j \neq \lambda_k$, the equation (3.5.53) follows from (3.5.57).

Next, to prove (3.5.54), we substitute (3.5.55) with $\phi = \mathbf{v}_j$ into (3.5.53) and obtain

$$\lambda_j \lambda_k (\mathbf{v}_j, \mathbf{M}\mathbf{w}_k) + \lambda_k^2 (\mathbf{v}_j, \mathbf{M}\mathbf{w}_k) + \lambda_k (\mathbf{v}_j, \mathbf{C}\mathbf{w}_k) = 0\,,$$

which implies (3.5.54) when $\lambda_k \neq 0$.

To prove relation (3.5.54) when $\lambda_k = 0$, we substitute $\phi = \mathbf{v}_j$ into the equation (3.5.55) and take (3.5.56) into account to obtain

$$0 = (\mathbf{v}_j, \mathbf{K}\mathbf{w}_k) = \lambda_j^2 (\mathbf{v}_j, \mathbf{M}\mathbf{w}_k) + \lambda_j (\mathbf{v}_j, \mathbf{C}\mathbf{w}_k)\,,$$

which implies (3.5.54), since $\lambda_j \neq \lambda_k = 0$.

We now show that the recent result on the orthogonality relations for the undamped gyroscopic quadratic operator pencil, established in [18], can be recovered from the above theorem.

Corollary 3.3 (Orthogonality of the Eigenfunctions of the Undamped Gyroscopic Quadratic Operator Pencil).

Consider the undamped gyroscopic quadratic operator pencil

$$\mathbf{P}(\lambda) \quad = \quad \lambda^2 \mathbf{M} + \lambda \mathbf{G} + \mathbf{K}\,, \tag{3.5.58}$$

where the operators $\mathbf{M}$ *and* $\mathbf{K}$ *are self-adjoint and positive definite, and the operator* $\mathbf{G}$ *is skew-symmetric; that is,* $\mathbf{G}^* = -\mathbf{G}$*. Let* $\lambda_1, \lambda_2, \ldots$ *be the eigenvalues of* $\mathbf{P}(\lambda)$ *and let* $\mathbf{w}_1, \mathbf{w}_2, \ldots$ *be the right eigenvectors of* $\mathbf{P}(\lambda)$*.*

If $\lambda_j \neq \lambda_k$*, then*

$$(\lambda_j \mathbf{v}_j, \lambda_k M \mathbf{w}_k) + (\mathbf{v}_j, K \mathbf{w}_k) \quad = \quad 0\,. \tag{3.5.59}$$

Proof. Since $\mathbf{G}$ is skew-symmetric, $(w, \mathbf{G}w) = 0$. Taking $\phi = w$, (3.4.49) reduces to

$$\lambda^2 (w, \mathbf{M}w) + (w, \mathbf{K}w) = 0\,.$$

Since both $\mathbf{M}$ and $\mathbf{K}$ are positive definite, λ^2 is a negative real number. Thus, λ is a purely imaginary number.

Since $\mathbf{P}(\lambda)$ is gyroscopic, $\mathbf{P}(\lambda)^* = \mathbf{P}(-\overline{\lambda})$. Thus, the equation (3.4.52) implies that (λ_j, w_j) is a right eigenpair of (3.5.58) if and only if $(-\overline{\lambda}_j, w_j)$ is a left eigenpair of (3.5.58). Equation (3.5.53) of Theorem 3.10 then proves Corollary 3.3, since the purely imaginary number λ_j is such that $\lambda_j = -\overline{\lambda}_j$ and, thus, $v_j = w_j$.

CHAPTER 4

EXISTING METHODS AND THEIR COMPUTATIONAL AND ENGINEERING DIFFICULTIES

In this chapter we briefly describe the following most common approaches that are used to solve the Problems 1.1 and 1.2 for the quadratic pencils and discuss their computational and engineering difficulties. We also briefly mention some of the engineering and computational difficulties associated with the two most common approaches for solving the eigenvalue and eigenstructure assignment problems, namely, solution via first-order reformulation and independent modal space control approach.

4.1 Eigenvalue Assignment for Quadratic Matrix Pencil via First-Order Reformulation

The *eigenvalue assignment problem* for the first-order control system

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t) \tag{4.1.1}$$

is the problem of finding a matrix F such that the spectrum of the matrix $A - BF$ is the arbitrary set $S = \{\lambda_1, \ldots, \lambda_n\}$, closed under complex conjugation. The matrix $A - BF$ is called the *closed-loop matrix*. The matrix F is called the *feedback matrix*. The eigenvalue assignment of the system (4.1.1) will simply be referred to as the eigenvalue assignment problem for the pair (A, B).

In case only p eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$ $(p < n)$ are to be altered using feedback, by reassigning them to $\{\mu_1, \ldots, \mu_p\}$ and keeping the remaining $n - p$ eigenvalues

$\lambda_{p+1}, \ldots, \lambda_n$ unaltered, the problem is called the *partial eigenvalue assignment problem* for the pair (A, B).

The eigenvalue assignment problem for a first-order control system has been well studied in control literature and there now exist excellent numerical methods for this problem (e.g., see Chapter 11 of [7]).

An obvious approach for solving the eigenvalue assignment problem for a quadratic pencil is to recast the problem in terms of a first-order control system. There are some computational difficulties with this approach.

The standard first-order reformulation of the pair $(P(\lambda), B)$ is

$$\dot{z}(t) = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix} u(t), \tag{4.1.2}$$

where $z(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$. In this case, the matrix M has to be inverted, and, if it is ill-conditioned, then the state matrix will not be computed accurately. Furthermore, all the exploitable properties such as definiteness, sparsity, bandedness, etc., of the coefficient matrices M, D, and K, usually offered by a practical problem, will be completely destroyed.

The nonstandard first-order reformulation, such as

$$\begin{pmatrix} K & 0 \\ 0 & -M \end{pmatrix} \dot{z}(t) = \begin{pmatrix} 0 & K \\ K & C \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ -B \end{pmatrix} u(t)\,,$$

is a descriptor system of the form $E\dot{z}(t) = Az(t) + \hat{B}u(t)$, and the eigenvalue assignment methods for the descriptor systems, especially when the matrix E is ill-conditioned, are not well developed. Furthermore, in this formulation, though the symmetry is preserved, the other exploitable properties mentioned above are lost.

4.2 Eigenvalue Assignment for Quadratic Matrix Pencil via Independent Modal Space Control

A second approach, popularly known in the engineering literature as the *independent modal space control* (IMSC) *approach* (see [1]), also suffers from some serious computational difficulties, and it is almost impossible to implement this approach in practice. The basic idea here is to decouple the symmetric problem (3.3.41) into a set of n independent problems, solve each of these independent problems separately, and then piece the individual solutions together to obtain a solution of the given problem. This is done using simultaneous diagonalization of the matrices M, C, and K as follows:

Let S be the matrix of eigenvectors of the linear pencil $K - \lambda M$ and let Λ_K be a diagonal matrix containing its eigenvalues. Then, since $M = M^T > 0$ and $K = K^T$, there exists a matrix S such that

$$S^T M S = I\,,\ S^T K S = \Lambda_K\,.$$

The same transforming matrix S also diagonalizes the matrix C, that is,

$$S^T C S = \Lambda_C$$

if and only if

$$KM^{-1}C \;=\; CM^{-1}K. \tag{4.2.3}$$

With the change in coordinates $x(t) = Sy(t)$, the equations

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) \;=\; Bu(t)\,,$$

$$u(t) \;=\; F^T\dot{x}(t) + G^T x(t)$$

then become

$$I\ddot{y}(t) + (\Lambda_C - S^T BF^T S)\dot{y}(t) + (\Lambda_K - S^T BG^T S)y(t) = 0\,.$$

These differential equations will decouple into n independent equations if and only if [1]

$$BF^T M^{-1}C = CM^{-1}BF^T \quad \text{and} \quad BG^T M^{-1}K = KM^{-1}BG^T. \qquad (4.2.4)$$

The commutativity relations (4.2.3) and (4.2.4) are almost impossible to satisfy in practice. Indeed, it is remarked by Inman: "This puts very stringent requirements on the locations and number of sensors and actuators" [1]. Furthermore, computing the matrices Λ_K, Λ_C, and S amounts to finding the complete spectrum and the associated eigenvectors of the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$.

Unfortunately, as stated in Section 3.2, numerical methods for finding the complete spectrum of the quadratic pencil are not well developed, especially for large and sparse matrices. The state-of-the-art techniques are capable of computing only a few extremal eigenvalues and eigenvectors (see [27, 29, 30, 41, 42, 43]).

The current engineering practice is to solve these problems using a small number of eigenvalues and the corresponding eigenvectors, hoping that the remaining large number of eigenvalues that are not to be reassigned remain invariant; that is, *spill-over* does not occur. Spill-over, however, may occur with such an ad hoc practice and can alter the eigenvalues to the point of destabilizing the system (see [14]).

4.3 Eigenstructure Assignment

Given the first-order control system

$$\dot{x}(t) = Ax(t) + Bu(t)\,,$$

the *eigenstructure assignment problem* is the problem of assigning both a given self-conjugate set of eigenvalues and the corresponding eigenvectors. Assigning the eigenvalues allows one to alter the stability characteristics of the system while assigning eigenvectors alters the transient response of the system. For a robust closed-loop solution, the eigenvectors are chosen either to be as orthogonal as possible or to have a specific modal structure (see [9, 15, 44, 45, 46, 47, 48] for details on choosing the eigenvectors).

In case only $p < n$ eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$ and the corresponding eigenvectors $\{x_1, \ldots, x_p\}$ are to be altered using feedback, by reassigning them to $\{\mu_1, \ldots, \mu_p\}$ and $x_{c1}, \ldots, x_{cp}$, respectively, and keeping the remaining $n-p$ eigenvalues $\lambda_{p+1}, \ldots, \lambda_n$ and eienvectors $x_{p+1}, \ldots, x_n$ unaltered, the problem is called the *partial eigenstructure assignment problem* for the first-order system.

The following theorem (Theorem 4.1) shows that the eigenstructure assignment problem is not always solvable; that is, a given set of eigenvectors may not be assignable. The theorem states the restrictions on the choice of eigenvectors.

Theorem 4.1 [49].

Let $\{\lambda_1, \ldots, \lambda_n\}$ be a self-conjugate set of distinct complex numbers. Then there exists a real $m \times n$ matrix F such that

$$(A - BF)x_{cj} = \lambda_j x_{cj}\,, \quad j = 1, 2, \ldots, n$$

if and only if for each j

(*i*) $\{x_{c1}, \ldots, x_{cn}\}$ *is a linearly independent set in* $\mathbb{C}^n$

(*ii*) $x_{cj} = \overline{x_{ck}}$ *when* $\lambda_j = \overline{\lambda_k}$

(*iii*) $x_{cj} \in \text{span}(N_{\lambda_j})$, *where* N_{λ_j} *is formed out of the first n rows of the matrix, whose columns form a basis for the nullspace of* $(\lambda_j I - A, B)$.

Proof. See [49].

4.3.1 Solving the Eigenstructure Assignment Problem of a Quadratic Matrix Pencil

Since the eigenstructure assignment is not always solvable when the control matrix B is given a priori, and on the other hand, in the vibration analysis of structural dynamic problems it is possible to choose the matrix B (see *e.g.* [1, 15]), we have developed in this dissertation a method for the partial eigenstructure assignment for a quadratic matrix pencil by choosing the matrix B. This method, like our proposed method for the partial eigenvalue assignment method, is again partial modal and direct. We shall present this method in Section 5.2.

4.4 Eigenvalue Assignment for Operator Pencil

The eigenvalue assignment problem for the operator pencil (Problem 1.3) can be solved either by *independent modal space approach* or by solving the problem for the discretized quadratic matrix pencil, obtained by finite element or finite difference method.

In the following we briefly state the independent modal space control approach for the partial eigenvalue assignment problem for the operator pencil and the possible computational and engineering difficulties.

4.4.1 Independent Modal Space Control for Distributed-Parameter System

Consider the forced distributed-parameter system of the form

$$\nu_{tt}(t,x) + L_1\,\nu_t(t,x) + L_2\,\nu(t,x) = h(t,x) \tag{4.4.5}$$

with appropriate boundary and initial conditions. Assume that L_1 and L_2 are self-adjoint positive definite operators, L_2 has a compact inverse, and L_1 shares the common set of eigenfunctions, $\{\phi_1, \phi_2, \ldots\}$, with L_2 (that is, L_1 and L_2 commute, which is essentially the condition (4.2.3) of Section 4.2 generalized to the operator settings). Then the control force of the form

$$h(t,x) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x) \tag{4.4.6}$$

can be computed using the *independent modal space control* (IMSC) *approach*, as follows:

First, we take the scalar product of (4.4.5) with ϕ_n and obtain

$$(\nu_{tt}, \phi_n) + (L_1\nu_t, \phi_n) + (L_2\nu, \phi) = (h, \phi_n)\,.$$

Substituting (4.4.6) into this equation, we obtain (see [1] for details)

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = h_n(t) \quad n = 1, 2, \ldots\,, \tag{4.4.7}$$

where $\lambda_n^{(i)}$ is the eigenvalues of L_i, $i = 1, 2$, corresponding to the eigenfunction ϕ_n and $a_n(t) = (\nu(t,x), \phi_n(x))$. These scalar equations in the functions $a_n(t)$ can be analyzed and solved separately for $h_n(t)$'s , and the control force $h(t,x)$ can be pieced together using (4.4.6).

Thus, to realize independent modal space control, the operator analogs of the conditions (4.2.3) and (4.2.4) have to be satisfied, which are almost impossible to happen in practice.

Furthermore, solving (4.4.7) amounts to finding the complete (infinite) spectrum and the associated eigenvectors of the pencil $\lambda^2 + \lambda L_1 + L_2$, while the state-of-the-art techniques are capable of computing only a few extremal eigenvalues and eigenvectors [43].

The current engineering practice is to solve these problems using a small number of eigenvalues and the corresponding eigenvectors, hoping that we still get a good approximation for the infinite sum in (4.4.6) and also that the remaining large number of eigenvalues that are not to be reassigned remain invariant (that is, *spill-over* does not occur). Spill-over, however, may occur with such an ad hoc practice and can alter the eigenvalues to the point of destabilizing the system (see [14]).

CHAPTER 5

PROPOSED SOLUTIONS FOR THE PARTIAL EIGENVALUE AND EIGENSTRUCTURE ASSIGNMENT PROBLEMS FOR THE QUADRATIC MATRIX PENCIL

In view of numerical difficulties (discussed at the end of the Section 3.1) associated with standard and generalized eigenvalue problems arising from the quadratic eigenvalue problem and the drawbacks associated with dealing with reformulated eigenvalue assignment problems outlined in Chapter 4, it is natural to wonder if solutions of the Problem 1.1 (partial eigenvalue assignment problem) and Problem 1.2 (partial eigenstructure assignment problem) can be obtained using only a partial knowledge of eigenvalues and eigenvectors of the quadratic pencil, without resorting to a first-order reformulation. A solution technique of this type will be called a *direct partial modal approach.* It is "direct" because the solution is obtained directly in the second-order setting without any types of reformulations. It is "partial modal" because only a part of the spectral data is needed for the solution.

In this chapter, we develop such a "direct and partial modal" approach for Problem 1.1 and Problem 1.2. These results are developed for a first-order problem first and then generalized to the quadratic matrix pencil. Several recent results on these problems are recovered as special cases. These include the results in [16] dealing with the single-input symmetric definite quadratic pencil, those in [19] dealing with the multi-input partial eigenvalue assignment for a symmetric quadratic pencil under the assumption that the stiffness matrix K is nonsingular, those in [18] dealing with the single-input partial eigenvalue assignment for an undamped gyroscopic pencil, and those in [17] dealing with the partial eigenstructure assignment for the symmetric definite quadratic pencil.

5.1 Partial Eigenvalue Assignment for the Quadratic Matrix Pencil

Let us recall that the eigenvalue assignment problem where only a small part of the spectrum has to be reassigned (usually the part that does not satisfy the design constraints) and the rest of the spectrum has to remain unaltered is called *partial eigenvalue assignment problem* (see Problem 1.1 in Chapter 1 for a formal definition). In this section we first establish the existence and uniqueness result for the partial eigenvalue assignment problem for the first-order system and then use it to derive an analogous result for the partial eigenvalue assignment problem for the quadratic pencil.

5.1.1 Existence and Uniqueness for the Eigenvalue Assignment Problem

First, we state a well known result on the existence and uniqueness of solution of the eigenvalue assignment problem. The notion of *controllability* is crucial to these results (see, for example, [7]).

Theorem 5.1 (Eigenvector Criterion of Controllability).
The first-order system (4.1.1) or, equivalently, the pair (A, B) *is* controllable *with respect to the eigenvalue* λ *of* A *if* $y^H B \neq 0$ *for all* $y \neq 0$ *such that* $y^H A = \lambda y^H$.

Definition 5.1 *The system (4.1.1) or matrix pair* (A, B) *is* partially controllable *with respect to the subset* $\{\lambda_1, \ldots, \lambda_p\}$ *of the spectrum of* A *if it is controllable with respect to each of the eigenvalues* λ_j, $j = 1, \ldots, p$.

Definition 5.2 *The system (4.1.1) or pair* (A, B) *is* completely controllable *if it is controllable with respect to every eigenvalue of* A.

Theorem 5.2 (Existence and Uniqueness for the First-Order Eigenvalue Assignment Problem).

The eigenvalue assignment problem for the pair (A, B) *is solvable for any arbitrary set* S *if and only if* (A, B) *is completely controllable. The solution is unique if and only if the system is a single-input system (that is, if* B *is a vector). In the multi-input case, there are infinitely many solutions, whenever a solution exists.*

Proof. The proof is available in any control theory text book, *e.g.* [7, 50, 51].

We now prove a similar result for the existence and uniqueness for the partial eigenvalue assignment problem. *This result and the proof are new.*

Theorem 5.3 (Existence and Uniqueness Results for the First-Order Partial Eigenvalue Assignment Problem).

Let $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_n)$ *be the diagonal matrix containing the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of* $A \in \mathbb{C}^{n \times n}$. *Assume that the sets* $\{\lambda_1, \ldots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \ldots, \lambda_n\}$ *are disjoint. Let the eigenvalues* $\lambda_1, \ldots, \lambda_p$ *to be changed to* $\mu_1, \ldots, \mu_p$ *and the remaining eigenvalues to remain invariant.*

Then the partial eigenvalue assignment problem for the pair (A, B) *is solvable for any choice of the closed-loop eigenvalues* $\mu_1, \ldots, \mu_p$ *if and only if the pair* (A, B) *is partially controllable with respect to the set* $\{\lambda_1, \ldots, \lambda_p\}$. *The solution is unique if and only if the system is a completely controllable single-input system. In the multi-input case, and in the single-input case when the system is not completely controllable, there are infinitely many solutions, whenever a solution exists.*

Proof. We first prove the *necessity*. Suppose the pair (A, B) is not controllable with respect to some λ_j, $1 \leq j \leq p$. Then there exists a vector $y \neq 0$ such that $y^H(A -$

$\lambda_j I) = 0$ and $y^H B = 0$. This means that for any F, we have $y^H(A - BF - \lambda_j I) = 0$, which implies that λ_j is an eigenvalue of $A - BF$ for every F, and thus λ_j cannot be reassigned.

Next we prove the *sufficiency*. Denote $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $\Lambda_2 = \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_n)$. Then we need to prove that there exists a feedback matrix F which assigns the eigenvalues in Λ_1 arbitrarily while keeping all the other eigenvalues unaltered.

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be, respectively, the right and left eigenvector matrices of A, and let $Y_1 = (y_1, \ldots, y_p)$. Since $Y^H X = I$ and $Y^H A X = \mathrm{diag}(\Lambda_1, \Lambda_2)$, then the partial controllability of the matrix pair (A, B) with respect to eigenvalues in Λ_1 implies the partial controllability of the pair $(\mathrm{diag}(\Lambda_1, \Lambda_2), Y^H B)$ with respect to the same eigenvalues. Therefore, the pair $(\Lambda_1, Y_1^H B)$ is completely controllable because $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} = \emptyset$.

By Theorem 5.2, there exists a feedback matrix Φ such that the closed-loop matrix $\Lambda_1 - Y_1^H B \Phi$ has the desired eigenvalues $\mu_1, \ldots, \mu_p$. Denote

$$F = \Phi Y_1^H . \tag{5.1.1}$$

Then the eigenvalues of the closed-loop matrix are exactly as required. This is seen as follows:

$$\begin{aligned} \{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_n\} &= \lambda\left(\mathrm{diag}(\Lambda_1, \Lambda_2) - Y^H B(\Phi, 0)\right) = \\ \lambda\left(Y^H\left(A - B((\Phi, 0)Y^H)\right)X\right) &= \lambda\left(A - B(\Phi Y_1^H)\right) . \end{aligned} \tag{5.1.2}$$

Uniqueness of the solution in the single-input case that is completely controllable and the existence of infinitely many solutions in the multi-input case follows directly from the Theorem 5.2.

To complete the proof we now show that infinitely many solutions to the partial eigenvalue assignment problem are possible when B is a vector (single-input case) and there exists an uncontrollable eigenvalue λ_k for some $k > p$ (that is, the associated k^{th} right eigenvector y_k is such that $y_k^H A = \lambda_k y_k^H$ and $y_k^H B = 0$).

Let F be a solution to the partial eigenvalue assignment problem. Denote the left and right eigenvectors of the closed-loop matrix $A_c = A - BF$ by Y_c and X_c. Clearly $y_k^H A_c = y_k^H (A - BF) = \lambda_k y_k$ and thus y_k is also the k^{th} column of Y_c. Let $F_\alpha = \alpha y_k^H$, where α is an arbitrary scalar. As in (5.1.2) we can show that the eigenvalues $\mu_1, \ldots, \mu_p,\ \lambda_{p+1}, \ldots, \lambda_{k-1},\ \lambda_{k+1}, \ldots, \lambda_n$ of A_c remain unchanged by the application of feedback F_α. Furthermore, the eigenvalue λ_k of A_c also remains unchanged by the feedback F_α, since the pair (A_c, B) is not controllable with respect to λ_k by the *necessity* part of this theorem. Thus

$$\lambda\,(\,A - BF\,) = \lambda\,(\,A_c\,) = \lambda\,(\,A_c - BF_\alpha\,) = \lambda\left(\,A - B(F + \alpha y_k^H)\,\right),$$

showing that if F is a solution, so is $F + \alpha y_k^H$ for an arbitrary α.

Since the *controllability* of the quadratic matrix pencil $P(\lambda)$ is defined in terms of the controllability of its first-order realization, Theorem 5.3 implies the following existence and uniqueness result for the quadratic matrix pencil $P(\lambda)$:

Corollary 5.1 (Existence and Uniqueness Results for the Partial Eigenvalue Assignment Problem for the Quadratic Pencil).
Let $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$, where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of the quadratic pencil $P(\lambda)$ that need to be reassigned and $\lambda_{p+1}, \ldots, \lambda_{2n}$ are the eigenvalues that are to remain invariant.

Then the partial eigenvalue assignment problem for the pair $(P(\lambda), B)$ is solvable for any choice of the closed-loop eigenvalues if and only if the pair $(P(\lambda), B)$ is *partially controllable with respect to $\lambda_1, \ldots, \lambda_p$. The solution is unique if and only if the system is completely controllable and is a single-input system. Otherwise, if the problem is solvable, there are infinitely many solutions.*

5.1.2 A Constructive Method for the Partial Eigenvalue Assignment of a First-Order System

The constructive proof of Theorem 5.3 yields a numerical method for constructing a feedback matrix F for the partial eigenvalue assignment problem for the pair (A, B). The major computational requirements are computation of a small number of eigenvalues and the corresponding eigenvectors of the matrix A and a solution of a small eigenvalue assignment problem, for which there now exist excellent numerical methods (see [7]).

The following method constructs a feedback matrix F such that

$$\lambda(A - BF) = \{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_n\}$$

assuming that

$$\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} = \emptyset$$

and that the pair (A, B) is partially controllable with respect to the set $\{\lambda_1, \ldots, \lambda_p\}$.

1. Compute the eigenvalues $\lambda_1, \ldots, \lambda_p$ that need to be reassigned and the corresponding left eigenvectors $y_1, \ldots, y_p$.

   Form $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$ and $Y_1 = (y_1, \ldots, y_p)$.

2. Compute a feedback matrix Φ of order $m \times p$ such that

$$\lambda\left(\Lambda_1 - Y_1^H B\Phi\right) = \{\mu_1, \ldots, \mu_p\}. \qquad (5.1.3)$$

3. Compute $F = \Phi Y_1^H$.

In [21], Saad has developed a projection method for solving the partial eigenvalue assignment problem for the pair (A, B). This method uses an orthogonal basis for the invariant subspace of A^T associated with the eigenvalues that need to be reassigned to project the problem to a smaller dimensional problem.

It is not clear if the Saad method can be extended to solve the partial eigenvalue assignment problem for the second-order pair $(P(\lambda), B)$, where $P(\lambda) = \lambda^2 M + \lambda C + K$. Recall that the standard first-order reformulation of the pair $(P(\lambda), B)$ is $(A, \hat{B})$, where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \text{ and } \hat{B} = \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix}.$$

It is possible to construct the basis for the invariant subspace of A^T from the left eigenvectors of $P(\lambda)$ without forming the A^T explicitly. However, since the matrix A^T is, in general, not a first-order representation of a quadratic pencil, it might not be possible to find a second-order analogue of the required projection of A^T.

On the other hand, the method we just proposed can be extended to the quadratic problem, as shown below. Before we do so, we first describe how to transform a complex conjugate set of eigenvalues and the corresponding set of eigenvectors to the real ones in the following lemma. This will be required to obtain real feedback matrices from the complex ones.

Lemma 5.1 (Transformation of the Self-Conjugate Pair (Λ, X) to the Real Pair (Λ_R, X_R)).
Let $\{\lambda_1, \ldots, \lambda_n\}$ be a set of complex numbers, closed under complex conjugation, and let the vectors $x_1, \ldots, x_n$ be such that $\overline{x_j} = x_k$ whenever $\overline{\lambda_j} = \lambda_k$. Then there exists

a nonsingular matrix T such that

$$T^{-1} = T^H\,,\ T\Lambda T^H = \Lambda_R\,,\ XT^H = X_R\,, \tag{5.1.4}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$*,* $X = (x_1, \ldots, x_n)$*, and both* Λ_R *and* X_R *are real matrices.*

Proof. Let

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Then the matrix S has the following properties:

$$S^{-1} = S^H, S \begin{pmatrix} x+iy & 0 \\ 0 & x-iy \end{pmatrix} S^H = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, (x+iy, x-iy)S^H = (\sqrt{2}x, \sqrt{2}y)\,.$$

Thus, the matrix $T = (t_{ij})$, defined by

$$\begin{cases} t_{jj} = t_{jk} = \frac{1}{\sqrt{2}}\,,\ t_{kj} = \frac{i}{\sqrt{2}}\ \ t_{kk} = -\frac{i}{\sqrt{2}}\,, & \text{if } \overline{\lambda_j} = \lambda_k \\ \qquad\qquad t_{jj} = 1\,, & \text{if } \lambda_j \text{ is real} \\ \qquad\qquad t_{ij} = 0\,, & \text{otherwise,} \end{cases} \tag{5.1.5}$$

will have the properties (5.1.4).

Based on Lemma 5.1, Theorem 3.1 that relates a quadratic eigenvalue problem to a standard eigenvalue problem and Theorem 3.8 on the orthogonality relations between the eigenvectors of a quadratic pencil, we can state the following result:

Theorem 5.4 (A Constructive Solution for the Partial Eigenvalue Assignment for the Quadratic Pencil).

Let $\{\lambda_1, \ldots, \lambda_{2n}\}$ *be the eigenvalues of the quadratic pencil* $P(\lambda) = \lambda^2 M + \lambda C + K$ *and let* $\{\mu_1, \ldots, \mu_p\}$ *be a given self-conjugate set of complex numbers. Assume* *(i)* $\{\lambda_1, \ldots, \lambda_p\}$ *is self-conjugate*

(ii) $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$

(iii) $(P(\lambda), B)$ *is partially controllable with respect to* $\{\lambda_1, \ldots, \lambda_p\}$.

Define the feedback matrices F_1 *and* F_2 *by*

$$F_1 = \Phi_R Y_{1R}^H M \text{ and } F_2 = \Phi_R(\Lambda_{1R} Y_{1R}^H M + Y_{1R}^H C)\,, \tag{5.1.6}$$

where Λ_{1R} *and* Y_{1R} *are defined from* $\Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$ *and* $Y_1 = (y_1, \ldots, y_p)$*, as in Lemma 5.1. Let* Φ_R *be a real feedback matrix such that*

$$\lambda\left(\Lambda_{1R} - (-Y_{1R}^H B)\Phi_R\right) = \{\mu_1, \ldots, \mu_p\}\,. \tag{5.1.7}$$

Then the closed-loop quadratic pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + K - BF_2$ *has the spectrum* $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

Proof. Recall from equation (4.1.2) in Section 4.1 that the feedback matrices F_1 and F_2 solve the quadratic eigenvalue problem if and only if the feedback matrix $F = (F_2, F_1)$ solves the first-order eigenvalue problem for the pair $(A, \hat{B})$, where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \text{ and } \hat{B} = \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix}.$$

From Theorem 3.1 we know that the matrix of eigenvectors $\hat{Y}_1$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_p$ of A is given by

$$\hat{Y}_1^H = \left(\Lambda_1 Y_1^H M + Y_1^H C,\; Y_1^H M\right). \tag{5.1.8}$$

Using Lemma 5.1 we can construct a matrix T such that $T^{-1} = T^H$, $T\Lambda_1 T^H = \Lambda_{1R}$, and $\hat{Y}_1 T^H = \hat{Y}_{1R}$ (note that the matrix $Y_{1R} = Y_1 T^H$ is also real).

From (5.1.8) and the equation (5.1.3), we then have

$$T(\Lambda_1 - \hat{Y}_1^H \hat{B}\Phi)T^{-1} = T(\Lambda_1 + Y_1^H B\Phi)T^{-1} =$$
$$(T\Lambda_1 T^H) + (TY_1^H)B(\Phi T^H) = \Lambda_{1R} + Y_{1R}^H B\Phi_R\,.$$

Thus, if we compute Φ_R from (5.1.7), then $\Phi = \Phi_R T$ satisfies (5.1.3).

Finally,

$$(F_2, F_1) = F = \Phi\hat{Y}_1^H = (\Phi_R T)\hat{Y}_1 =$$
$$\Phi_R\left((T\Lambda_1 T^H)(TY_1^H)M + (TY_1^H)C,\ (TY_1^H)M\right) =$$
$$\Phi_R\left(\Lambda_{1R} Y_{1R}^H M + Y_{1R}^H C,\ Y_{1R}^H M\right)\,.$$

Thus, (5.1.6) is verified and the proof is complete.

Remark 5.1 *Theorem 5.4 is important from the practical applications viewpoint because it reduces the partial eigenvalue assignment problem for the quadratic pencil involving large $n \times n$ matrices to a much smaller standard eigenvalue assignment problem of order $p < n$, where p is the number of eigenvalues that are to be changed. Such a reduction can be accomplished by means of a moderate amount of spectral data - just the right eigenvectors corresponding to the few eigenvalues that are to be changed are needed. And these few eigenvalues and eigenvectors can be computed with the state-of-the-art iterative methods.*

Corollary 5.2 *If, in addition to the conditions of Theorem 5.4, it is assumed that none of the eigenvalue $\lambda_1, \ldots, \lambda_p$ is zero, then (5.1.6) is simplified to*

$$F_1 = \Phi_R Y_{1R}^H M \text{ and } F_2 = -\Phi_R \Lambda_{1R}^{-1} Y_{1R}^H K\,. \tag{5.1.9}$$

Proof. Indeed, using the notations of Theorem 5.4, we simplify the formula for F_2 as follows:

$$F_2 = \Phi_R(\Lambda_{1R}Y_{1R}^H M + Y_{1R}^H C) = \Phi_R T(\Lambda_1 Y_1^H M + Y_1^H C) =$$
$$\Phi_R T\Lambda_1^{-1}(\Lambda_1^2 Y_1^H M + \Lambda_1 Y_1^H C) = \Phi_R T\Lambda_1^{-1} Y_1^H K = \Phi_R \Lambda_{1R}^{-1} Y_{1R}^H K\,.$$

Corollary 5.2 says that in the case when $\lambda_1, \ldots, \lambda_p$ are nonzero, the knowledge of the matrix $C = D + G$ is not explicitly needed to compute the feedback matrices F_1 and F_2. This is important when partial eigenvalue assignment is to be performed to control a symmetric or undamped gyroscopic system (see Chapter 2 for details), since the *damping* matrix D is the most imprecise part of the model (see [1]). On the other hand, eigenvalues (also called the *poles* or *modes* of a system) and right eigenvectors (called *mode shapes*) could often be measured for such models and converted to left eigenvectors using Corollary 3.1 (symmetric system) or 3.2 (undamped gyroscopic systems) without the explicit knowledge of D.

Example 2 (An illustrative example).

We consider the quadratic matrix pencil

$$P(\lambda) = \lambda^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix}$$

having the open-loop eigenvalues

$$\lambda_1 = -4.8341\,,\ \lambda_2 = -1.4842\,,\ \lambda_{3,4} = -2 \pm 2.2361i \text{ and } \lambda_{5,6} = 0.15915 \pm 0.90052i.$$

Let the control matrix be

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then the pair $(P(\lambda), B)$ is controllable with respect to only the eigenvalues λ_1, λ_2, λ_5, and λ_6.

To reassign the three most unstable eigenvalues λ_2, λ_5, and λ_6 to $\mu_{1,2} = -2 \pm i$ and $\mu_3 = -2$, we use Corollary 5.2 as follows:

Step 1. We use Lemma 5.1 to convert

$$\begin{aligned} \Lambda_1 &= \operatorname{diag}(-1.4842, 0.15915 - 0.90052, 0.15915 + 0.90052)\,, \\ Y_1 &= \begin{pmatrix} -0.48587 & -0.29409 - 0.42023i & -0.29409 + 0.42023i \\ -0.75964 & 0.84048 & 0.84048 \\ -0.4323 & -0.14853 + 0.091917i & -0.14853 - 0.091917i \end{pmatrix} \end{aligned}$$

to real matrices

$$\Lambda_{1R} = \begin{pmatrix} -1.4842 & 0 & 0 \\ 0 & 0.1591 & -0.9005 \\ 0 & 0.9005 & 0.1591 \end{pmatrix}, Y_{1R} = \begin{pmatrix} -0.4859 & -0.4159 & -0.5943 \\ -0.7596 & 1.1886 & 0 \\ -0.4323 & -0.2100 & 0.1300 \end{pmatrix},$$

respectively, using the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & 0.7071 \\ 0 & 0.7071i & -0.7071i \end{pmatrix}.$$

Step 2. We solve the eigenvalue assignment problem for the pair $(\Lambda_{1R}, -Y_{1R}^H B)$

using a standard eigenvalue assignment method (function `place` from the MATLAB Control Toolbox) and obtain a real feedback matrix

$$\Phi_R = \begin{pmatrix} -0.7794 & 1.0652 & 5.3866 \\ 0.2153 & -1.1820 & 0.7890 \end{pmatrix},$$

such that the spectrum of the matrix $\Lambda_{1R}+Y_{1R}^H B\Phi_R$ is the set $\{-2+i, -2-i, -2\}$.

Step 3. Using (5.1.9), we compute

$$F_1 = \Phi_R Y_{1R}^H M = \begin{pmatrix} -3.2655 & 1.8582 & 0.8134 \\ -0.0819 & -1.5685 & 0.2578 \end{pmatrix}$$

and

$$F_2 = -\Phi_R \Lambda_{1R}^{-1} Y_{1R}^H K = \begin{pmatrix} 1.5780 & 7.8485 & 3.0386 \\ -7.1252 & -7.1428 & -8.4425 \end{pmatrix}.$$

Verification. It is easy to verify that the eigenvalues of the closed-loop quadratic pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ are $\{-4.8341, -2, -2 \pm i, -2 \pm 2.2361i\}$.

5.1.3 A Parameterization Approach for Partial Eigenvalue Assignment

In this section, we develop a parametric approach to the partial eigenvalue assignment problems for both the first-order pair (A, B) and the quadratic pair $(P(\lambda), B)$. We then describe a numerical algorithm for the quadratic problem.

We remark that developing parametric solutions to these problems is useful in that one can then think of solving some other important variation of the problems, such as the robust partial eigenvalue assignment problem, by exploiting freedom of these parameters.

We make the following assumptions that will simplify the proofs of our theorems for the rest of the chapter. Justification for each of these assumptions is also stated.

Assumption 5.5 *The control matrix B has full rank.*

Justification: Indeed, if the $n \times m$ matrix B has rank $m_1 < m$, then it admits the economy-size QR decomposition $B = QR$, where R is an $m_1 \times m$ matrix of full rank. Suppose that we have performed partial eigenvalue assignment with the full-rank matrix Q (instead of B) and obtained the feedback matrix K. Then $QK = BF = (QR)F$ and we recover the feedback matrix F for use with the original control matrix B solving the underdetermined linear system $K = RF$ in the least-square sense.

Note that if the full-rank matrix B is close to the rank-deficient matrix; that is, if the absolute values of some diagonal entries of R are less than certain tolerance, then elimination of such "almost linearly dependent" parts of B through the economy-size QR decomposition might result in a better feedback matrix.

Example 3 (Rank deficient control matrix) *Consider the control matrix B and the feedback matrix F defined by*

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{pmatrix}.$$

The economy-size QR decomposition of B is

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} (\sqrt{3}, \sqrt{3}) = QR.$$

Therefore, $BF = B_{\text{new}}F_{\text{new}}$*, where*

$$B_{\text{new}} = Q = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

is a full rank matrix and

$$F_{\text{new}} = RF = (-3\sqrt{3}, -3\sqrt{3}, -3\sqrt{3}).$$

Now suppose that in order to satisfy Assumption 5.5, the feedback control problem was solved with full-rank B_{new} *instead of rank-deficient* B *and a feedback matrix*

$$K = (1, 2, 3)$$

was obtained. To get the equivalent 2×3 *feedback matrix* F_{old} *corresponding to the control matrix* B*, we then solve the linear system*

$$K = (1, 2, 3) = (\sqrt{3}, \sqrt{3})F_{\text{old}} = RF_{\text{old}}$$

to obtain

$$F_{\text{old}} = R^{\dagger}K = \begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} (1, 2, 3) = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

It is easily verified that $B_{\text{new}}K = BF_{\text{old}}$*.*

Assumption 5.6 *The sets* $\{\lambda_1, \ldots, \lambda_n\}$ *and* $\{\mu_1, \ldots, \mu_p\}$ *are closed under complex conjugation and disjoint.*

Justification: The closeness under complex conjugation of the above sets is necessary to guarantee that the closed-loop eigenvalues are self-conjugate, since any closed-loop system designed with the feedback that has physical sense (that is, real feedback) must have self-conjugate spectrum.

If $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} \neq \emptyset$, it means that some open-loop eigenvalues that we have selected to reassign in fact would not move. In this case, we should renumber the open-loop eigenvalues in such a way that the eigenvalues that would remain unaltered would go last and the number p of the eigenvalues to be reassigned will be decreased. This way we obtain the partial eigenvalue assignment problems with $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} = \emptyset$.

Designing a closed-loop system such that $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} \neq \emptyset$ is generally considered a "bad practice" in engineering. Systems with such artificially created multiple eigenvalues are usually less robust compared to the systems designed with slightly perturbed $\mu_1, \ldots, \mu_p$ because multiple eigenvalues are usually very sensitive to perturbations.

The following theorem gives a parametric solution to the first-order partial eigenvalue assignment problem.

Theorem 5.7 (Parametric Solution to First-Order Partial Eigenvalue Assignment Problem).

Let the Assumptions 5.5 and 5.6 hold and let the pair (A, B) be partially controllable with respect to $\{\lambda_1, \ldots, \lambda_p\}$. Assume further that the closed-loop matrix has a complete set of eigenvectors. Let $\Gamma = (\gamma_1, \ldots, \gamma_p)$ be a matrix such that

$$\gamma_j = \overline{\gamma_k} \text{ whenever } \mu_j = \overline{\mu_k}\,. \tag{5.1.10}$$

Set $\Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$ and $\Lambda_{c1} = \operatorname{diag}(\mu_1, \ldots, \mu_p)$. Let Z_1 be a unique nonsingular *solution of the Sylvester equation*

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B\Gamma \,. \tag{5.1.11}$$

Then the real feedback matrix F given by

$$F = \Phi Y_1^H \,, \tag{5.1.12}$$

where Φ satisfies the linear system

$$\Phi Z_1 = \Gamma \,, \tag{5.1.13}$$

solves the partial eigenvalue assignment problem for the pair (A, B).

Conversely, if there exists a real feedback matrix F of the form (5.1.12) that solves the partial eigenvalue assignment problem for the pair (A, B), then the matrix Φ can be constructed satisfying (5.1.10) through (5.1.13).

Proof. First, we prove the "converse part" of the Theorem. Let a matrix F of the form (5.1.12) solve the partial eigenvalue assignment problem. Denote by $X_{c1} = (x_{c1}, \ldots, x_{cp})$ the matrix of right eigenvectors of the closed-loop pencil corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$. Define the matrix $\Gamma = FX_{c1}$. Then the following equation is obviously satisfied:

$$AX_{c1} - X_{c1}\Lambda_{c1} = B\Gamma \,. \tag{5.1.14}$$

Multiplying this equation to the left by the Y_1^H and defining $Z_1 = Y_1^H X_{c1}$, we obtain (5.1.11).

From (5.1.12) and (5.1.14), we have

$$0 = (A - B\Phi Y_1^H)X_{c1} - X_{c1}\Lambda_{c1} = B\Gamma - B\Phi Y_1^H X_{c1} = B(\Gamma - \Phi Z_1)\,. \quad (5.1.15)$$

Since B has linearly independent columns, (5.1.15) is equivalent to (5.1.13).

Finally, if $\mu_j = \overline{\mu_k}$, then $x_{cj} = \overline{x_{ck}}$, where $X_{c1} = (x_{c1}, \ldots, x_{cp})$. Since F is real and the j^{th} column of Γ is $\gamma_j = Fx_{cj}$, we get $\gamma_j = \overline{\gamma_k}$, proving (5.1.10).

Now we will prove the theorem in the other direction. Let Γ be chosen to satisfy (5.1.10) and (5.1.11). Since $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} = \emptyset$, then Φ is also uniquely defined by (5.1.11) and (5.1.13).

Using (3.3.32), we note that for any Φ with $F = \Phi Y_1^H$ we have

$$(A - BF)X_2 = AX_2 - B\Phi(Y_1^H X_2) = X_2\Lambda_2\,, \quad (5.1.16)$$

where $\Lambda_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_n)$ and X_2 are the right eigenvectors of A corresponding to the eigenvalues $\lambda_{p+1}, \ldots, \lambda_n$. Thus, both the eigenvalues $\lambda_{p+1}, \ldots, \lambda_n$ and the associated right eigenvectors $X_2 = (x_{p+1}, \ldots, x_n)$ of the closed-loop system are the same as those of the open-loop system.

It thus remains to be shown that with our above choice of Φ, the set $\{\mu_1, \ldots, \mu_p\}$ is also in the spectrum of $A - BF$ and the matrix F is real.

Since the set $\{\mu_1, \ldots, \mu_p\}$ and the spectrum of A are disjoint, the Sylvester equation (5.1.14) has a unique solution (see [7] for details), which we denote by X_{c1}. Multiplying the equation (5.1.14) by Y_1^H and noting that $Y_1^H A = \Lambda_1 Y_1^H$, we obtain

$$\Lambda_1(Y_1^H X_{c1}) - (Y_1^H X_{c1})\Lambda_{c1} = Y_1^H B\Gamma\,. \quad (5.1.17)$$

Thus, $Y_1^H X_{c1}$ and Z_1 satisfy the same Sylvester equation. Since this Sylvester equation has a unique solution (because spectra of Λ_1 and Λ_{c1} are disjoint), we have

$$Z_1 \quad = \quad Y_1^H X_{c1} \,. \tag{5.1.18}$$

Using (5.1.13) and (5.1.18), we obtain

$$(A - BF)X_{c1} - X_{c1}\Lambda_{c1} = AX_{c1} - X_{c1}\Lambda_{c1} - B\Phi Y_1^H X_{c1} = B(\Gamma - \Phi(Y_1^H X_{c1})) = 0,$$

which shows that the set $\{\mu_1, \ldots, \mu_p\}$ is in the spectrum of $A - BF$.

To complete the proof of the theorem, we must show that F is real.

Since the set $\{\mu_1, \ldots, \mu_p\}$ is closed under complex conjugation, there exists a permutation matrix T_c such that $\overline{\Lambda_{c1}} = T_c^T \Lambda_{c1} T_c$. Then (5.1.10) implies that $\overline{\Gamma} = \Gamma T_c$. Similarly, there exists a permutation matrix T such that $\overline{\Lambda_1} = T^T \Lambda_1 T$, $\overline{X_1} = X_1 T$ and $\overline{Y_1} = Y_1 T$. Conjugating the equation (5.1.11), we get

$$(T^T \Lambda_1 T)\overline{Z_1} - \overline{Z_1}(T_c^T \Lambda_{c1} T_c) = (T^T Y_1^H) B (\Gamma T_c) \,. \tag{5.1.19}$$

Clearly $\overline{Z_1} = T^T Z_1 T_c$, since such $\overline{Z_1}$ satisfies the equation (5.1.19) as $T_c^T = T_c^{-1}$. Again, conjugating (5.1.13), we get

$$\overline{\Phi}(T^T Z_1 T_c) = \Gamma T_c \,,$$

which implies that $\overline{\Phi} = \Phi T$.

Therefore,

$$\overline{F} = (\Phi T)(T^T Y_1^H) = F \,,$$

showing that the obtained feedback matrix F is real.

Remark 5.2 *Substituting the expression of* Γ *from (5.1.13) into (5.1.11), we obtain*

$$(\Lambda_1 - Y_1^H B\Phi)Z_1 = Z_1\Lambda_{c1}\,, \tag{5.1.20}$$

which shows that Z_1 *is the eigenvector matrix for* $\Lambda_1 - Y_1^H B\Phi$*. From (5.1.20) it then follows that the nonsingularity of* Z_1 *is equivalent to the linear independence of the eigenvectors of the closed-loop matrix* $A - BF$*.*

This observation is important because it is well known that the sensitivity of the eigenvalues of the closed-loop matrix is related to the conditioning of the eigenvector matrix.

We now show how Theorem 5.7 can be extended to the solution of the partial eigenvalue assignment problem for the quadratic pencil.

Theorem 5.8 (Parametric Solution to the Partial Eigenvalue Assignment Problem for the Quadratic Pencil).
Let the matrix B *have full rank. Let the scalars* $\mu_1, \ldots, \mu_p$ *and the eigenvalues* $\lambda_1, \ldots, \lambda_{2n}$ *of the open-loop quadratic pencil* $P(\lambda) = \lambda^2 M + \lambda C + K$ *be such that the sets* $\{\lambda_1, \ldots, \lambda_p\}$*,* $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$*, and* $\{\mu_1, \ldots, \mu_p\}$ *are disjoint and each set is closed under complex conjugation. Let the pair* $(P(\lambda), B)$ *be partially controllable with respect to* $\{\lambda_1, \ldots, \lambda_p\}$*. Let* $\Gamma = (\gamma_1, \ldots, \gamma_p)$ *be a matrix such that*

$$\gamma_j = \overline{\gamma_k} \text{ whenever } \mu_j = \overline{\mu_k}\,. \tag{5.1.21}$$

Set $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ *and* $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$ *. Let* Z_1 *be the unique nonsingular solution of the Sylvester equation*

$$\Lambda_1 Z_1 - Z_1\Lambda_{c1} = -Y_1^H B\,\Gamma\,, \tag{5.1.22}$$

where Y_1 denotes the matrix of left eigenvectors corresponding to $\lambda_1, \ldots, \lambda_p$. Then for the real feedback matrices F_1 and F_2 given by

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H C)\,, \tag{5.1.23}$$

where Φ satisfies the linear system

$$\Phi Z_1 = \Gamma\,, \tag{5.1.24}$$

solve the partial eigenvalue assignment problem for the pair $(P(\lambda), B)$.

Conversely, if there exist real feedback matrices F_1 and F_2 of the form (5.1.23) that solves the partial eigenvalue assignment problem for the pair $(P(\lambda), B)$, then the matrix Φ can be constructed satisfying (5.1.21) through (5.1.24).

Proof. As in the proof of Theorem 5.4, we reduce the quadratic eigenvalue assignment problem to the partial eigenvalue assignment problem for the pair (A, B) defined by

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \text{ and } \hat{B} = \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix}.$$

We then apply Theorem 5.7 to get the real feedback matrix F, which solves the problem for the pair (A, B), and finally recover the feedback matrices F_1 and F_2 of the quadratic problem as $F = (F_2, F_1)$.

To prove that the pair of matrices F_1 and F_2 defined by (5.1.21) through (5.1.24) is the solution of the quadratic problem, we note that these expressions reduce to, respectively, (5.1.10) through (5.1.13) of Theorem 5.7 when we set $F = (F_2, F_1)$. This is because by Theorem 3.1 the matrix of left eigenvectors $\hat{Y}_1$ of A corresponding to eigenvalues $\lambda_1, \ldots, \lambda_p$ of A is given by

$$\hat{Y}_1^H = \left(\Lambda_1 Y_1^H M + Y_1^H C,\, Y_1^H M\right)\,,$$

and

$$\hat{Y}_1^H \hat{B} = \left(\Lambda_1 Y_1^H M + Y_1^H C,\, Y_1^H M\right) \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix} = -Y_1^H B\,.$$

Remark 5.1 and Corollary 5.2 are also valid here. Of special importance is the Corollary 5.2 on avoiding the use of damping matrix. Here is the explicit statement, analogous to Corollary 5.2.

Remark 5.3 (Avoiding the Explicit Usage of the Damping Matrix in the Parametric Solution).
If, in addition to the conditions of Theorem 5.8, it is known that none of the eigenvalues $\lambda_1, \ldots, \lambda_p$ is zero, then (5.1.23) can be simplified to

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = -\Phi \Lambda_1^{-1} Y_1^H K\,. \tag{5.1.25}$$

Thus, in this case, explicit knowledge of the damping matrix is not needed in computing the feedback matrices. Since, as stated before, in practice it is hard to estimate damping, obtaining the feedback matrices without the explicit knowledge of the damping matrix is quite useful in practical applications.

Furthermore, note that the restriction that none of the eigenvalues $\lambda_1, \ldots, \lambda_p$ is zero can be easily removed by a shifting procedure, as described in [16].

Based on the Theorem 5.8, we now state the following algorithm:

Algorithm 5.9 (An Algorithm for the Multi-input Partial Pole Placement Problem for the Quadratic Matrix Pencil).

Inputs:

(a) The $n \times n$ *matrices* M, C, *and* K.

(b) The $n \times m$ *control matrix* B.

(c) The set $\{\mu_1, \ldots, \mu_p\}$, *closed under complex conjugation.*

(d) The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ *of the open-loop spectrum* $\{\lambda_1, \ldots, \lambda_{2n}\}$ *and the associated right eigenvector set* $\{y_1, \ldots, y_p\}$.

Outputs:

The feedback matrices F_1 *and* F_2 *such that the spectrum of the closed-loop pencil* $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ *is* $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

Assumptions:

(a) M *is nonsingular and* B *has full rank.*

(b) The quadratic pencil $P(\lambda)$ *with control matrix* B *is partially controllable with respect to the eigenvalues* $\lambda_1, \ldots, \lambda_p$.

(c) The sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$, *and* $\{\mu_1, \ldots, \mu_p\}$ *are disjoint.*

Step 1. *Form* $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $Y_1 = (y_1, \ldots, y_p)$, *and* $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$.

Step 2. *Choose arbitrary* $m \times 1$ *vectors* $\gamma_1, \ldots, \gamma_p$ *in such a way that* $\overline{\mu_j} = \mu_k$ *implies* $\overline{\gamma_j} = \gamma_k$ *and form* $\Gamma = (\gamma_1, \ldots, \gamma_p)$.

Step 3. *Find the unique solution* Z_1 *of the Sylvester equation*

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B \Gamma \,.$$

If Z_1 is ill-conditioned, then return to Step 2 and select different $\gamma_1, \ldots, \gamma_p$.

Step 4. *Solve $\Phi Z_1 = \Gamma$ for Φ.*

Step 5. *If none of the $\lambda_1, \ldots, \lambda_p$ is zero, form*

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = -\Phi \Lambda_1^{-1} Y_1^H K \,,$$

otherwise form

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H C)\,.$$

Remark 5.4 (Some Distinctive Feature of Algorithm 5.9).

The most distinctive feature of the algorithm is that it computes the solution of a large partial eigenvalue assignment problem by solving a small linear algebraic system and by using only the few eigenvalues of the associated large quadratic pencil that need to be reassigned and the corresponding right eigenvectors. Thus, the algorithm is readily applicable to control dangerous vibration in a structure, where only a small part of the spectrum needs to be reassigned and the rest is to remain unchanged. Furthermore, one can take complete advantage of the sparsity, symmetry, definiteness, etc., of the matrices M, C, and K in computing F_1 and F_2.

Example 4 (An illustrative example).

We consider the same quadratic pencil as in Example 2,

$$P(\lambda) = \lambda^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix}$$

having the open-loop eigenvalues

$$\lambda_1 = -4.8341\,,\ \lambda_2 = -1.4842\,,\ \lambda_{3,4} = -2 \pm 2.2361i\,,\ \text{and}\ \lambda_{5,6} = 0.15915 \pm 0.90052i$$

and the control matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then the pair $(P(\lambda), B)$ is controllable with respect to only the eigenvalues λ_1, λ_2, λ_5, and λ_6.

To reassign the three most unstable eigenvalues λ_2, λ_5, and λ_6 to $\mu_{1,2} = -2 \pm i$ and $\mu_3 = -2$, we use Algorithm 5.9 as follows:

Step 1. We form

$$\begin{aligned} \Lambda_1 &= \operatorname{diag}(-1.4842, 0.15915 - 0.90052i, 0.15915 + 0.90052i), \\ Y_1 &= \begin{pmatrix} -0.48587 & -0.29409 - 0.42023i & -0.29409 + 0.42023i \\ -0.75964 & 0.84048 & 0.84048 \\ -0.4323 & -0.14853 + 0.091917i & -0.14853 - 0.091917i \end{pmatrix} \end{aligned}$$

and

$$\Lambda_{c1} = \operatorname{diag}(-2 - i, -2 + i, -2)\,.$$

Step 2. We choose arbitrary

$$\Gamma = \begin{pmatrix} 0.70886 - 0.60881i & 0.70886 + 0.60881i & 0.57559 \\ 0.051498 + 0.35244i & 0.051498 - 0.35244i & -0.81774 \end{pmatrix}.$$

Step 3. Solving the Sylvester equation $\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B\,\Gamma$, we obtain

$$Z_1 = \begin{pmatrix} 0.13408 - 0.31438i & 0.13408 + 0.31438i & -0.66215 \\ -0.05836 - 0.35539i & 0.069126 + 0.143i & 0.37773 + 0.045516i \\ 0.069126 - 0.143i & -0.05836 + 0.35539i & 0.37773 - 0.045516i \end{pmatrix}.$$

Condition number of Z_1 is 5.2448 and, since Z_1 is well-conditioned, we proceed to Step 4.

Step 4. Solving $\Phi Z_1 = \Gamma$ we get

$$\Phi = \begin{pmatrix} -0.61709 & 0.66607 + 3.6933i & 0.66607 - 3.6933i \\ 0.21049 & -0.87936 + 0.15417i & -0.87936 - 0.15417i \end{pmatrix}.$$

Step 5. We form

$$F_1 = \begin{pmatrix} -3.196 & 1.5884 & 0.74786 \\ 0.28538 & -1.6381 & 0.19857 \end{pmatrix}$$

and

$$F_2 = \begin{pmatrix} 0.70415 & 6.6231 & 1.5162 \\ -6.8006 & -7.3842 & -7.8925 \end{pmatrix}.$$

Verification. It is easy to verify that the eigenvalues of the closed-loop quadratic pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ are

$$\{-4.8341, -2, -2 \pm i, -2 \pm 2.2361i\}.$$

5.1.4 Recovery of Recent Results

As stated in the Introduction of this chapter, several papers [5, 16, 17, 18, 19, 52], etc., dealing with special cases of the quadratic partial eigenvalue assignment problem have been published in the last few years. These results now can be recovered as special cases of Theorem 5.8. We show below how the result in [16] on the single-input problem for the symmetric positive definite quadratic matrix pencil can be recovered. The derivations of the other results are analogous.

Corollary 5.3 (Datta, Elhay, and Ram [16]).

Let

$$MX\Lambda^2 + DX\Lambda + KX = 0,$$

where $X \in \mathbb{C}^{n\times 2n}$ *and* $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \in \mathbb{C}^{2n\times 2n}$, λ_i *distinct, be the eigendecomposition of the open-loop symmetric definite quadratic pencil*

$$P(\lambda) = \lambda^2 M + \lambda D + K, \text{ where } M = M^T > 0, D = D^T, K = K^T.$$

Let X *and* Λ *be partitioned as* $X = (X_1, X_2)$ *and* $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$, *where* $X_1 = (x_1, \ldots, x_p)$ *and* $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$. *Let* f, g *be chosen as*

$$f = MX_1\Lambda_1\beta\,, \quad g = -KX_1\beta\,, \quad \beta \in \mathbb{C}^{p\times 1} \tag{5.1.26}$$

with the components β_j *of* β *given by*

$$\beta_j = \frac{1}{b^T x_j}\frac{\mu_j - \lambda_j}{\lambda_j}\prod_{\substack{i=1\\ i\neq j}}^{m}\frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}\,. \tag{5.1.27}$$

Then, the closed-loop pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D - bf^T) + (K - bg^T)$$

has the spectrum $\{\mu_1, \mu_2, \ldots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}\}$.

Proof. Since the quadratic pencil $P(\lambda)$ is symmetric, by Corollary 3.1, we have $Y_1^H = X_1^T$, where X_1 and Y_1 are respectively left and right eigenvectors of $P(\lambda)$ corresponding to the same eigenvalues. Let's define $\beta^T = \Phi\Lambda^{-1}$. Then from (5.1.23) of Theorem 5.8, we have

$$F_1 = \beta^T \Lambda_1 X_1^T M \text{ and } F_2 = -\beta^T X_1^T K\,.$$

Since $f = F_1^T$ and $g = F_2^T$, we obtain (5.1.26).

Again, choosing $\Gamma = (1, 1, \ldots, 1)$ in equation (5.1.24) and with $\beta^T = \Phi\Lambda^{-1}$, we have

$$\beta^T \Lambda_1 Z_1 = (1, 1, \ldots, 1). \tag{5.1.28}$$

Since for (5.1.28), the jk^{th} element z_{jk} of the matrix Z_1 is given by

$$z_{jk} = \frac{x_j^T b}{\mu_k - \lambda_j}\,,$$

we easily see that β given by (5.1.27) satisfies (5.1.28).

5.2 Partial Eigenstructure Assignment for the Quadratic Matrix Pencil

Let us recall that the eigenstructure assignment problem, where only a small part of the set of eigenvalues and their corresponding eigenvectors have to be reassigned (usually the part that does not satisfy the design constraints) and the rest of the eigenvalues and eigenvectors have to remain unaltered, is called the *partial eigenstructure assignment problem* (see Problem 1.2 in Chapter 1 for a formal definition).

In this section we develop a "partial modal approach" for the partial eigenstructure assignment problem of the quadratic pencil $P(\lambda)$. As in Section 5.1.3, we restrict ourselves to the case when Assumption 5.6 holds true.

As before, we first derive conditions for the existence of solution of the partial eigenstructure assignment problem for the first-order system and then show how the conditions for existence of solution of the partial eigenstructure assignment problem of quadratic pencil can be derived from that of the first-order system.

Theorem 5.10 (Solution to the First-Order Partial Eigenstructure Assignment Problem).
Let the scalars $\mu_1, \ldots, \mu_p$ and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the open-loop matrix A be such that the sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_n\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint and individually closed under complex conjugation. Let the "eigenvectors to be assigned" $x_{c1}, \ldots, x_{cp}$ and the "eigenvectors to be kept invariant" $x_{p+1}, \ldots, x_n$ form a two-fold complete system.

Define the matrix

$$Z_1 \quad = \quad Y_1^H X_{c1}, \tag{5.2.29}$$

where $Y_1 = (y_1, \ldots, y_p)$ is a matrix of left eigenvectors of A corresponding to $\lambda_1, \ldots, \lambda_p$ and $X_{c1} = (x_{c1}, \ldots, x_{cp})$. If Z_1 is nonsingular, then

(i) The pair of matrices (B, F) given by

$$B \quad = \quad AX_{c1} - X_{c1}\Lambda_{c1} \tag{5.2.30}$$

and

$$F \quad = \quad Z_1^{-1} Y_1^H \tag{5.2.31}$$

constitutes a (possibly complex) solution to the partial eigenstructure assignment problem.

(ii) A solution with real B_R *and* F_R *is obtained from* B *and* F *as*

$$B_R = BT_c^H \text{ and } F_R = T_c F\,, \tag{5.2.32}$$

where the matrix T_c *is such that*

$$T_c^{-1} = T_c^H \text{ and the matrices } T_c \Lambda_{c1} T_c^H \text{ and } X_{c1} T^H \text{ are real}$$

(see Lemma 5.1 for an explicit construction of T_c*).*

Conversely, if there exists a pair of real matrices B *and* F *that solves the eigenstructure assignment problem, then the matrix* Z_1 *defined by (5.2.29) is nonsingular.*

Proof. We prove the "converse" part of the theorem first. Let the matrices B and F in the form (5.2.30) and (5.2.31), respectively, constitute a solution. Without loss of generality we can assume that B has full rank (see justification of Assumption 5.5 for details).

Since the matrix Y of left eigenvectors of A is nonsingular, then the matrix F can be decomposed as

$$F = \Phi Y_1^H + \Phi_2 Y_2^H, \text{ where } Y = (Y_1, Y_2) = (y_1, \ldots, y_p; y_{p+1}, \ldots, y_n). \tag{5.2.33}$$

Because the closed-loop system $A - BF$ should retain the eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$ and their corresponding eigenvectors $X_2 = (x_{p+1}, \ldots, x_n)$, and since $Y_1^H X_2 = 0$ and $Y_2^H X_2 = I$ by Theorem 3.6, we obtain

$$0 = (A - BF)X_2 - X_2 \Lambda_2 =$$

$$(AX_2 - X_2\Lambda_2) - B(\Phi Y_1^H X_2 + \Phi_2 Y_2^H X_2) = -B\Phi_2\,. \tag{5.2.34}$$

Since B has full rank, $\Phi_2 = 0$, and thus F has the form of (5.1.12). Then we obtain $Z_1 = Y_1^H X_{c1}$ in a way similar to the way the equations (5.1.17) and (5.1.18) were derived in Theorem 5.7. Furthermore, for (5.1.20), we see that Z_1 is nonsingular.

Next, we prove the theorem in the other direction. Since Z_1 is nonsingular, let B and F be defined by (5.2.30) and (5.2.31), respectively. Since F has a form (5.2.33) with $\Phi_2 = 0$, the equation (5.2.34) shows that both the eigenvalues $\lambda_{p+1}, \ldots, \lambda_n$ and their left eigenvectors $x_{p+1}, \ldots, x_n$ of the closed-loop system are the same as those of the open-loop system. Furthermore, since

$$(A - BF)X_{c1} - X_{c1}\Lambda_{c1} = (AX_{c1} - X_{c1}\Lambda_{c1}) - BFX_{c1} = B - BZ_1^{-1}Y_1^H X_{c1} = 0$$

the set $\{\mu_1, \ldots, \mu_p\}$ is in the spectrum of $A - BF$, and $\{x_{c1}, \ldots, x_{cp}\}$ is the corresponding eigenvector set. This completes the proof of part (i).

Unfortunately, matrices B and F might be complex. To obtain the real matrices B_R and F_R such that $BF = B_R F_R$, we use Lemma 5.1 that explicitly constructs the matrices T and T_c such that

$$T^{-1} = T^H \text{ and } T_c^{-1} = T_c^H, \tag{5.2.35}$$

and $T\Lambda_1 T^H$, TY_1^H, $T_c\Lambda_{c1}T_c^H$, and $X_{c1}T^H$ are real matrices.

Using (5.2.35) we have

$$BT_c^H = A(X_{c1}T_c^H) - (X_{c1}T_c^H)(T_c\Lambda_{c1}T_c^H)\,,$$

which shows that the matrix BT_c^H is real. Similarly, for Z_1 we get that $TZ_1T_c^H$ is real and thus

$$T_c Z_1^{-1} T^H = (T Z_1 T_c^H)^{-1} \tag{5.2.36}$$

is also real. Using (5.2.31), (5.2.35), and (5.2.36) it is now easy to see that $F_R = T_c F$ is also real and the proof is complete.

We now show how Theorem 5.10 can be extended to the solution of the partial eigenstructure assignment problem for the quadratic pencil.

Theorem 5.11 (Solution to the Partial Eigenstructure Assignment Problem for the Quadratic Pencil).

Let the scalars $\mu_1, \ldots, \mu_p$ and the eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ of the open-loop quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ be such that the sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint and individually closed under complex conjugation. Let the "eigenvectors to be assigned" $x_{c1}, \ldots, x_{c2}$ and the "eigenvectors to be kept invariant" $x_{p+1}, \ldots, x_{2n}$ form a two-fold complete system (that is, we require that the first-order realization of a closed-loop system would have a complete set of eigenvectors).

If the matrix

$$Z_1 \quad = \quad \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1} \tag{5.2.37}$$

is nonsingular, then

(i) The triplet of matrices (B, F_1, F_2) given by

$$B = M X_{c1} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}, \tag{5.2.38}$$

$$F_1 = Z_1^{-1} Y_1^H M, \text{ and } F_2 = Z_1^{-1} (\Lambda_1 Y_1^H M + Y_1^H C) \tag{5.2.39}$$

constitutes a (possibly complex) solution to the partial eigenstructure assignment problem.

(ii) A solution with real B_R, F_{1R}, and F_{2R} is obtained from B, F_1, and F_2 as

$$B_R = BT_c^H\ ,\ F_{1R} = T_cF_1\ ,\ \textit{and}\ F_{2R} = T_cF_2\ , \tag{5.2.40}$$

where the matrix T_c, such that

$$T_c^{-1} = T_c^H \ \textit{and the matrices}\ T_c\Lambda_{c1}T_c^H \ \textit{and}\ X_{c1}T^H \ \textit{are real,}$$

is explicitly constructed in Lemma 5.1.

Conversely, if there exists a triplet of real matrices (B, F_1, F_2) that solves the eigenstructure assignment problem, then the matrix Z_1 defined by (5.2.37) is nonsingular.

Proof. From the first-order reformulation of the pair $(P(\lambda), B)$ we know that the triplet (B, F_1, F_2) solves the partial eigenstructure assignment problem for the quadratic pencil $P(\lambda)$ if and only if the pair of matrices $(\hat{B}, \hat{F})$ solves the partial eigenstructure assignment problem for the matrix A, where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}\ ,\ \hat{B} = \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix} \text{ and } \hat{F} = (F_2, F_1)\,.$$

We then apply Theorem 5.10 to get the solution matrices $\hat{B}$ and $\hat{F}$ and recover the matrices B, F_1, and F_2 of the quadratic problem as follows:

$$Z_1 = \hat{Y}_1^H\hat{X}_{c1} = (\Lambda_1 Y_1^H M + Y_1^H C, Y_1^H M)\begin{pmatrix} X_{c1} \\ X_{c1}\Lambda_{c1} \end{pmatrix} =$$
$$\Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1}\Lambda_{c1} + Y_1^H C X_{c1}\,,$$

which verifies (5.2.37). Again,

$$\hat{B} = A\hat{X}_{c1} - \hat{X}_{c1}\Lambda_{c1} = \begin{pmatrix} X_{c1}\Lambda_{c1} \\ -M^{-1}(KX_{c1} + CX_{c1}\Lambda_{c1}) \end{pmatrix} - \begin{pmatrix} X_{c1}\Lambda_{c1} \\ X_{c1}\Lambda_{c1}^{2} \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ -M^{-1}(KX_{c1} + CX_{c1}\Lambda_{c1} + MX_{c1}\Lambda_{c1}^{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ -M^{-1}B \end{pmatrix},$$

which verifies (5.2.38), and

$$(F_2, F_1) = \hat{F} = Z_1^{-1}(\Lambda_1 Y_1^H M + Y_1^H C, Y_1^H M)\,,$$

which verifies (5.2.39). The proof of part (i) is complete.

Just as in Theorem 5.10, the matrices B, F_1, and F_2 might be complex. To get the real matrices B_R, F_{1R}, and F_{2R} such that $BF_1 = B_R F_{1R}$ and $BF_2 = B_R F_{2R}$, we use Lemma 5.1 that explicitly constructs the matrices T and T_c such that

$$T^{-1} = T^H \text{ and } T_c^{-1} = T_c^H, \tag{5.2.41}$$

and $T\Lambda_1 T^H$, TY_1^H, $T_c\Lambda_{c1}T_c^H$, and $X_{c1}T^H$ are real matrices.

Using (5.2.41) we get

$$BT_c^H = M(X_{c1}T_c^H)(T_c\Lambda_{c1}T_c^H)^2 + C(X_{c1}T_c^H)(T_c\Lambda_{c1}T_c^H) + K(X_{c1}T_c^H),$$

which shows that the matrix BT_c^H is real. Similarly, for Z_1 we get that $TZ_1T_c^H$ is real and thus

$$T_c Z_1^{-1} T^H = (TZ_1T_c^H)^{-1} \tag{5.2.42}$$

is also real. Using (5.2.39), (5.2.41), and (5.2.42) it is easy to see that both T_cF_1 and T_cF_2 are real and the proof is complete.

Remark 5.5 *If the control matrix obtained in Theorem 5.11 is rank deficient then an approach based on economy-size QR decomposition can be used to reduce the number of control inputs (see justification of Assumption 5.5 for details).*

Remark 5.6 *Theorem 5.11 can be used to recover the solution to partial eigenstructure assignment problem for the symmetric definite quadratic pencil proposed in [18] and [53].*

Based on the Theorem 5.11 we can state the following algorithm:

Algorithm 5.12 (An Algorithm for the Multi-input Partial Eigenstructure Assignment Problem for a Quadratic Matrix Pencil).

Inputs:

(a) The $n \times n$ matrices M, C, and K.

(b) The set of scalars $\{\mu_1, \ldots, \mu_p\}$ and the set of vectors $\{x_{c1}, \ldots, x_{cp}\}$, both closed under complex conjugation.

(c) The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_{2n}\}$ and the associated right eigenvector set $\{y_1, \ldots, y_p\}$.

Outputs:

The $n \times m$ control matrix B and the feedback matrices F_1 and F_2 such that the spectrum of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ is $\{\mu_1$, $\ldots$, μ_p; λ_{p+1}, $\ldots$, $\lambda_{2n}\}$ with the eigevector matrix $X_c = (x_{c1}, \ldots, x_{cp}; x_{p+1}, \ldots, x_{2n})$.

Assumptions:

(a) M is nonsingular.

(b) The sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint.

Step 1. *Form $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $Y_1 = (y_1, \ldots, y_p)$, $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$, and $X_{c1} = (x_{c1}, \ldots, x_{cp})$.*

Step 2. *Form the matrix*

$$Z_1 = \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1}.$$

Stop if Z_1 is singular and conclude that the eigenstructure assignment with the given sets of eigenvalues and eigenvectors is not possible.

Step 3. *Using Lemma 5.1, form the matrix T_c such that $T_c \Lambda_{c1} T_c^H$ is a real matrix.*

Step 4. *Form*

$$\begin{aligned} B &= (M X_{c1} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}) T_c^H, \\ F_1 &= T_c Z_1^{-1} Y_1^H M, \text{ and} \\ F_2 &= T_c Z_1^{-1} (\Lambda_1 Y_1^H M + Y_1^H C) \end{aligned}$$

by solving the appropriate linear systems.

Remark 5.7 *The most distinctive feature of the algorithm is that it computes the solution of a large partial eigenstructure assignment problem by solving a small linear algebraic system and by using only the few eigenvalues of the large quadratic pencil that need to be reassigned and the associated right eigenvectors. This allows the algorithm to be readily applicable to control dangerous vibration in a structure, where* *only a small part of the spectrum needs to be reassigned and the rest is to remain unchanged.*

Example 5 (An illustrative example).

We consider the same quadratic pencil as in Example 2,

$$P(\lambda) = \lambda^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix},$$

having the open-loop eigenvalues

$$\lambda_1 = -4.8341\,,\ \lambda_2 = -1.4842\,,\ \lambda_{3,4} = -2 \pm 2.2361i\,,\ \text{and } \lambda_{5,6} = 0.15915 \pm 0.90052i.$$

To reassign the pair of unstable eigenvalues λ_5 and λ_6 to $\mu_{1,2} = -2 \pm i$ and assign their eigenvectors to $x_{c1,2} = (\pm i, 0, 0)^T$, respectively, we use Algorithm 5.12 as follows:

Step 1. We form

$$\begin{aligned} \Lambda_1 &= \mathrm{diag}(0.15915 - 0.90052, 0.15915 + 0.90052), \\ Y_1 &= \begin{pmatrix} -0.29409 - 0.42023i & -0.29409 + 0.42023i \\ 0.84048 & 0.84048 \\ -0.14853 + 0.091917i & -0.14853 - 0.091917i \end{pmatrix} \end{aligned}$$

and

$$\Lambda_{c1} = \mathrm{diag}(-2 - i, -2 + i)\,,\ X_{c1} = \begin{pmatrix} i & -i \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Step 2. We form

$$Z_1 = \begin{pmatrix} -0.20557 + 4.4079i & -0.38261 - 3.5674i \\ -0.38261 + 3.5674i & -0.20557 - 4.4079i \end{pmatrix}.$$

Condition number of Z_1 is 5.2448 and, since Z_1 is nonsingular, we proceed to Step 3.

Step 3. Using Lemma 5.1 we obtain the matrix

$$T_c = \begin{pmatrix} 0.70711 & 0.70711 \\ 0.70711i & -0.70711i \end{pmatrix}.$$

Step 5. We form

$$\begin{aligned} B &= \begin{pmatrix} -4.2426 & 11.314 \\ 5.6569 & -9.8995 \\ 0 & 0 \end{pmatrix} \\ F_1 &= \begin{pmatrix} 0.70711 & 0.031889 & -0.16274 \\ 0 & 0.15139 & -0.03834 \end{pmatrix} \end{aligned}$$

and

$$F_2 = \begin{pmatrix} 1.4142 & 0.28507 & 1.8613 \\ 0.70711 & 0.68564 & 0.75083 \end{pmatrix}.$$

Verification. It is easy to verify that the eigenvalues of the closed-loop quadratic pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ are

$$\{-4.8341, -1.4842, -2 \pm i, -2 \pm 2.2361i\}$$

and the eigenvectors, corresponding to eigenvalues $-2 \pm i$, are $\begin{pmatrix} \pm i \\ 0 \\ 0 \end{pmatrix}$, respectively.

CHAPTER 6

PARTIAL EIGENVALUE ASSIGNMENT FOR THE QUADRATIC OPERATOR PENCIL: PROPOSED PARAMETRIC SOLUTION

Recall from Chapter 2 that the natural models of the vibrating structures such as beams, buildings, bridges, highways, large space structures, etc., are the systems of partial differential equations (also called *distributed-parameter systems*) of the form

$$\mathbf{M}\frac{\partial^2 \nu(t,x)}{\partial t^2} + \mathbf{C}\frac{\partial \nu(t,x)}{\partial t} + \mathbf{K}\nu(t,x) \quad = \quad 0, \tag{6.0.1}$$

where $\nu(t,\cdot)$ belongs to some Hilbert space $\mathbb{H}$ with appropriate scalar product (ϕ,ψ) for all $t_0 \leq t \leq t_{\text{final}}$.

In this chapter we show how to obtain a direct and partial modal solution of Problem 1.3 in its own natural settings of distributed-parameter systems (6.0.1) with the controlling forces of the form

$$\sum_{k=1}^{m}\left((\mathbf{f}_{1k}(x), \frac{\partial \nu(t,x)}{\partial t}) + (\mathbf{f}_{2k}(x), \nu(t,x))\right)\mathbf{b}_k(x), \tag{6.0.2}$$

where the functions $\mathbf{b}_1(x), \ldots, \mathbf{b}_m(x)$ are the *control functions*, and functions $\mathbf{f}_{1k}, \mathbf{f}_{2k} \in \mathbb{H}$, $k = 1, \ldots, m$, are the *velocity* and *position feedback functions*, respectively.

6.1 Parametric Solution of the Partial Eigenvalue Assignment Problem

In this section we develop Theorem 6.3, which gives a family of parametric feedback matrices that solve the given partial eigenvalue assignment problem. Then in Section 6.1.1 we show how the recent result [18] on modal solutions for the partial eigenvalue assignment problem for undamped gyroscopic quadratic operator pencil can be recovered from this theorem. Numerical Algorithm 6.4, which generalizes the known algorithms, follow from Theorem 6.3 in a straightforward way.

First, without loss of generality we make the following assumptions that will simplify the proofs of our theorems in the rest of the chapter (see Section 5.1.3 for justification).

Assumption 6.1 *The control functions* $\mathbf{b}_1, \ldots, \mathbf{b}_m$ *are linearly independent.*

Assumption 6.2 *The sets* $\{\lambda_1, \lambda_2, \ldots\}$ *and* $\{\mu_1, \ldots, \mu_p\}$ *are closed under complex conjugation and they are disjoint sets.*

The following theorem investigates the parameterization of the feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ using the parameter matrix Γ.

Theorem 6.3 (Parametric Solution to the Partial Eigenvalue Assignment Problem for the Quadratic Operator Pencil).
Let the Assumptions 6.1 and 6.2 hold and the pair $(\mathbf{P}(\lambda), \mathbf{B})$ *be partially controllable with respect to* $\{\lambda_1, \ldots, \lambda_p\}$*. Let* $\Gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{C}^{m \times p}$ *be a matrix such that*

$$\gamma_j = \overline{\gamma_k} \text{ whenever } \mu_j = \overline{\mu_k}\,. \tag{6.1.3}$$

Let Z_1 be the unique nonsingular solution of the Sylvester equation

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (\mathbf{v}_1, \mathbf{b}_1) & \cdots & (\mathbf{v}_1, \mathbf{b}_m) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_p, \mathbf{b}_1) & \cdots & (\mathbf{v}_p, \mathbf{b}_m) \end{pmatrix} \Gamma \,, \tag{6.1.4}$$

where $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$. Let the matrix $\Phi = (\phi_{kj}) \in \mathbb{C}^{m \times p}$ be the solution of the linear system

$$\Phi Z_1 = \Gamma \,. \tag{6.1.5}$$

Then the real feedback functions $\mathbf{f}_{1k}$ and $\mathbf{f}_{2k}$ for $k = 1, 2, \ldots, m$ are given by

$$\mathbf{f}_{1k} = \sum_{j=1}^{p} \overline{\phi}_{kj} \mathbf{M}^* \mathbf{v}_j \quad \text{and} \quad \mathbf{f}_{2k} = \sum_{j=1}^{p} \overline{\phi}_{kj} (\overline{\lambda}_j \mathbf{M}^* \mathbf{v}_j + \mathbf{C}^* \mathbf{v}_j) \tag{6.1.6}$$

solves the partial eigenvalue assignment problem for the pair $(\mathbf{P}(\lambda), \mathbf{B})$.

Conversely, if there exist real feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ of the form (6.1.6) that constitute the solution of the partial eigenvalue assignment problem for the pair $(\mathbf{P}(\lambda), \mathbf{B})$, then the matrix Φ in (6.1.6) can be constructed satisfying (6.1.3) through (6.1.5).

Proof. First, we prove the converse part of the theorem. Let the feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ of the form (6.1.6) be such that they solve the partial eigenvalue assignment problem. Denote by $\mathbf{w}_{c1}, \ldots, \mathbf{w}_{cp}$ the right eigenvectors of the closed-loop pencil corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$. Define the matrix

$$\Gamma = (\gamma_{jk}) \in \mathbb{C}^{m \times p}, \text{ where } \gamma_{jk} = \mu_k (\mathbf{f}_{1j}, \mathbf{w}_{ck}) + (\mathbf{f}_{2j}, \mathbf{w}_{ck}) \,. \tag{6.1.7}$$

Then the following equations are obviously satisfied for all $k = 1, \ldots, r$:

$$\mu_k^2 \mathbf{M}\mathbf{w}_{ck} + \mu_k \mathbf{C}\mathbf{w}_{ck} + \mathbf{K}\mathbf{w}_{ck} = \sum_{j=1}^{m} \mathbf{b}_j \gamma_{jk}\,. \tag{6.1.8}$$

Taking now the scalar product with $\mathbf{v}_r$ on both sides of (6.1.8) we obtain

$$\mathbf{P}(\mu_k)\mathbf{w}_{ck} = \mu_k^2(\mathbf{v}_r, \mathbf{M}\mathbf{w}_{ck}) + \mu_k(\mathbf{v}_r, \mathbf{C}\mathbf{w}_{ck}) + (\mathbf{v}_r, \mathbf{K}\mathbf{w}_{ck}) = \sum_{j=1}^{m}(\mathbf{v}_r, \mathbf{b}_j)\gamma_{jk} \tag{6.1.9}$$

for each $r = 1, \ldots, p$. Noting that

$$(\mathbf{v}_r, \mathbf{K}\mathbf{w}_{ck}) = -\lambda_r^2(\mathbf{v}_r, \mathbf{M}\mathbf{w}_{ck}) - \lambda_r(\mathbf{v}_r, \mathbf{C}\mathbf{w}_{ck})$$

and adding and subtracting $\lambda_r \mu_k(\mathbf{v}_r, M\mathbf{w}_{ck})$ to (6.1.9) and regrouping, we have

$$\begin{aligned} &\mu_k\left[\mu_k(\mathbf{v}_r, \mathbf{M}\mathbf{w}_{ck}) + (\mathbf{v}_r, \mathbf{C}\mathbf{w}_{ck}) + \lambda_r(\mathbf{v}_r, M\mathbf{w}_{ck})\right] - \\ &\quad \lambda_r\left[\mu_k(\mathbf{v}_r, \mathbf{M}\mathbf{w}_{ck}) + (\mathbf{v}_r, \mathbf{C}\mathbf{w}_{ck}) + \lambda_r(\mathbf{v}_r, M\mathbf{w}_{ck})\right] = \sum_{j=1}^{m}(\mathbf{v}_r, \mathbf{b}_j)\gamma_{jk}\,. \end{aligned} \tag{6.1.10}$$

Denoting

$$Z_1 = (z_{rk}) \in \mathbb{C}^{p\times p} \text{ with } z_{rk} = (\lambda_r + \mu_k)(\mathbf{v}_r, \mathbf{M}\mathbf{w}_{ck}) + (\mathbf{v}_r, \mathbf{C}\mathbf{w}_{ck})\,, \tag{6.1.11}$$

the equation (6.1.10) becomes (6.1.4).

From (6.1.6), (6.1.8), and (6.1.11), we have for each $k = 1, \ldots, p$

$$\begin{aligned} 0 = \mathbf{P}_c(\mu_k)\mathbf{w}_{ck} &= \mathbf{P}(\mu_k)\mathbf{w}_{ck} - \sum_{r=1}^{m}\mathbf{b}_r\left(\mu_k(\mathbf{f}_{1r}, \mathbf{w}_{ck}) + (\mathbf{f}_{2r}, \mathbf{w}_{ck})\right) = \\ &\sum_{r=1}^{m}\mathbf{b}_r\gamma_{rk} - \sum_{r=1}^{m}\mathbf{b}_r\sum_{j=1}^{p}\phi_{rj}\Big((\mu_k + \lambda_j)(\mathbf{v}_j, \mathbf{M}\mathbf{w}_{ck}) + (\mathbf{v}_j, \mathbf{C}\mathbf{w}_{cj})\Big) = \\ &\sum_{r=1}^{m}\mathbf{b}_r\left(\gamma_{rk} - \sum_{j=1}^{p}\phi_{rj}z_{jk}\right)\,. \end{aligned} \tag{6.1.12}$$

Since B has linearly independent columns, (6.1.12) is equivalent to (6.1.5).

Finally, if $\mu_j = \overline{\mu_k}$, then $\mathbf{w}_{cj} = \overline{\mathbf{w}_{ck}}$. Since all $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ are real, (6.1.7) implies that $\gamma_{jr} = \overline{\gamma}_{kr}$ for any r, proving (6.1.3).

Now we will prove the theorem in the other direction. Let Γ be chosen to satisfy (6.1.3) and (6.1.4). Since $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} = \emptyset$, the solution matrix Z_1 of the matrix equation (6.1.4) is unique. Since Z_1 is nonsingular, Φ is uniquely defined by (6.1.5), for a given Γ.

Using (3.5.54) we note that for any Φ with $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ given in (6.1.6), we have for all $k = p+1, p+2, \ldots$

$$\mathbf{P}_c(\lambda_k)\mathbf{w}_{ck} = \mathbf{P}(\lambda_k)\mathbf{w}_{ck} - \sum_{r=1}^{m} \mathbf{b}_r \left(\lambda_k(\mathbf{f}_{1r}, \mathbf{w}_{ck}) + (\mathbf{f}_{2r}, \mathbf{w}_{ck})\right) =$$
$$0 - \sum_{r=1}^{m} \mathbf{b}_r \sum_{j=1}^{p} \phi_{rj} \Big((\lambda_k + \lambda_j)(\mathbf{v}_j, \mathbf{M}\mathbf{w}_{ck}) + (\mathbf{v}_j, \mathbf{C}\mathbf{w}_{cj})\Big) = 0\,, \qquad (6.1.13)$$

where $\mathbf{w}_{cp+1}, \mathbf{w}_{cp+2}, \ldots$ are the right eigenvectors of $\mathbf{P}(\lambda)$ corresponding to the eigenvalues $\lambda_{p+1}, \lambda_{p+2}, \ldots$. Thus, both the eigenvalues $\lambda_{p+1}, \lambda_{p+2}, \ldots$ and the associated right eigenvectors $\mathbf{w}_{cp+1}, \mathbf{w}_{cp+2}, \ldots$ of the closed-loop operator pencil $\mathbf{P}_c(\lambda)$ are the same as those of the open-loop operator pencil $\mathbf{P}(\lambda)$.

It thus remains to be shown that with our above choice of Φ, the operator pencil $\mathbf{P}_c(\lambda)$ has the set $\mu_1, \ldots, \mu_p$ in its spectrum and the control functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ are real functions.

Since the set $\{\mu_1, \ldots, \mu_p\}$ and the spectrum of A are disjoint, the equation (6.1.8) has the unique solution for all $k = 1, \ldots, p$, which we denote by $\mathbf{w}_{c1}, \ldots, \mathbf{w}_{cm}$. Repeating the steps in the proof of the "converse" direction, we establish (6.1.10). Thus, each rk^{th} element of Z_1 and $(\lambda_r + \mu_k)(\mathbf{v}_r, \mathbf{M}\mathbf{w}_{ck}) + (\mathbf{v}_r, \mathbf{C}\mathbf{w}_{ck})$ satisfies the same Sylvester equation. Since this Sylvester equation has a unique solution (because the spectra of Λ_1 and Λ_{c1} are disjoint), we conclude that Z_1 satisfies (6.1.11).

Using (6.1.5) and (6.1.11), we conclude that (6.1.12) holds, which shows that the set $\{\mu_1, \dots, \mu_p\}$ is in the spectrum of $\mathbf{P}_c(\lambda)$.

To complete the proof of the theorem, we must show that F is real.

Since the set $\{\mu_1, \dots, \mu_p\}$ is closed under complex conjugation, there exists a permutation matrix T_c such that $\overline{\Lambda_{c1}} = T_c^T \Lambda_{c1} T_c$. Then (6.1.3) implies that $\overline{\Gamma} = \Gamma T_c$. Similarly, there exists a permutation matrix T such that $\overline{\Lambda_1} = T^T \Lambda_1 T$, $\overline{\mathbf{W}_1} = \mathbf{W}_1 T$ and $\overline{\mathbf{V}_1} = \mathbf{V}_1 T$, where $\mathbf{W}_1 = (\mathbf{w}_1, \dots, \mathbf{w}_p)$ and $\mathbf{V}_1 = (\mathbf{v}_1, \dots, \mathbf{v}_p)$. Similarly, proceeding as in the last part of the proof of Theorem 5.8, we can show that $\overline{Z_1} = T^T Z_1 T_c$, $\overline{\Phi} = \Phi T$, and that all $\mathbf{f}_{11}, \dots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \dots, \mathbf{f}_{2m}$ are real functions.

Based on the Theorem 6.3 we state the following algorithm:

Algorithm 6.4 (Parametric Solution to the Partial Eigenvalue Assignment Problem for the Quadratic Operator Pencil).

Inputs:

(a) The differential operators $\mathbf{M}$*,* $\mathbf{C}$*, and* $\mathbf{K}$*.*

(b) The m *control functions* $\mathbf{b}_1, \dots, \mathbf{b}_m$*.*

(c) The set of scalars $\{\mu_1, \dots, \mu_p\}$*, closed under complex conjugation.*

(d) The self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$ *of the open-loop spectrum* $\{\lambda_1, \lambda_2, \dots\}$ *and the associated right eigenvector set* $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$*.*

Outputs:

The feedback functions $\mathbf{f}_{11}, \dots, \mathbf{f}_{1m}$ *and* $\mathbf{f}_{21}, \dots, \mathbf{f}_{2m}$ *such that the spectrum of the closed-loop operator pencil*

$$\mathbf{P}_c(\lambda)\phi = \lambda^2 \mathbf{M}\phi + \lambda \left(\mathbf{C}\phi - \sum_{r=1}^{m} (\mathbf{f}_{1r}, \phi)\mathbf{b}_r \right) + \left(\mathbf{K}\phi - \sum_{r=1}^{m} (\mathbf{f}_{2r}, \phi)\mathbf{b}_r \right)$$

is $\{\mu_1,\ \ldots,\ \mu_p;\ \lambda_{p+1},\ \lambda_{p+2},\ldots\}$.

Assumptions:

(a) The control functions $\mathbf{b}_1,\ldots,\mathbf{b}_m$ *are linearly independent.*

(b) The open-loop quadratic operator pencil $\mathbf{P}(\lambda)=\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}$ *with control functions* $\mathbf{b}_1,\ldots,\mathbf{b}_m$ *is partially controllable with respect to the eigenvalues* $\lambda_1,\ \ldots,\ \lambda_p$.

(c) The sets $\{\lambda_1,\ldots,\lambda_p\}$, $\{\lambda_{p+1},\lambda_{p+1},\ldots\}$, *and* $\{\mu_1,\ldots,\mu_p\}$ *are disjoint.*

Step 1. *Form* $\Lambda_1=\mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\mathbf{V}_1=(\mathbf{v}_1,\ldots,\mathbf{v}_p)$, *and* $\Lambda_{c1}=\mathrm{diag}(\mu_1,\ldots,\mu_p)$.

Step 2. *Choose arbitrary* $m\times 1$ *vectors* $\gamma_1,\ldots,\gamma_p$ *in such a way that* $\overline{\mu_j}=\mu_k$ *implies* $\overline{\gamma_j}=\gamma_k$ *and form* $\Gamma=(\gamma_1,\ldots,\gamma_p)$.

Step 3. *Solve the following Sylvester equation for* Z_1*:*

$$\Lambda_1 Z_1 - Z_1\Lambda_{c1} = \begin{pmatrix} (\mathbf{v}_1,\mathbf{b}_1) & \cdots & (\mathbf{v}_1,\mathbf{b}_m) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_p,\mathbf{b}_1) & \cdots & (\mathbf{v}_p,\mathbf{b}_m) \end{pmatrix}\Gamma\,.$$

If Z_1 *is ill-conditioned, then return to Step 2 and select different* $\gamma_1,\ldots,\gamma_p$ *.*

Step 4. *Solve* $\Phi Z_1=\Gamma$ *for* Φ.

Step 5. *If none of the* $\lambda_1,\ldots,\lambda_p$ *is zero, form for all* $k=1,\ldots,m$

$$\mathbf{f}_{1k}=\sum_{j=1}^{p}\overline{\phi}_{kj}\mathbf{M}^*\mathbf{v}_j \quad \text{and} \quad \mathbf{f}_{2k}=-\sum_{j=1}^{p}(\overline{\phi}_{kj}/\overline{\lambda}_j)\mathbf{K}^*\mathbf{v}_j\,,$$

otherwise form for all $k=1,\ldots,m$,

$$\mathbf{f}_{1k}=\sum_{j=1}^{p}\overline{\phi}_{kj}\mathbf{M}^*\mathbf{v}_j \quad \text{and} \quad \mathbf{f}_{2k}=\sum_{j=1}^{p}\overline{\phi}_{kj}(\overline{\lambda}_j\mathbf{M}^*\mathbf{v}_j+\mathbf{C}^*\mathbf{v}_j)\,.$$

An example for the Algorithm 6.4 is worked out in Section 7.2.

6.1.1 Recovery of Recent Result on the Partial Eigenvalue Assignment of the Quadratic Operator Pencil

As stated in the introduction of this chapter, several papers [18, 20], etc., dealing with special cases of the quadratic eigenvalue assignment problem for operator pencils have been published in the last few years. These results can be recovered as special cases of Theorem 6.3. We show below how the result in [18] on the single-input problem for an undamped gyroscopic quadratic operator pencil can be recovered. The derivations of the other result are analogous.

Corollary 6.1 (Datta, Ram, and Sarkissian [18])

Consider an undamped gyroscopic quadratic operator pencil

$$\mathbf{P}(\lambda) \quad = \quad \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}\,, \tag{6.1.14}$$

where the operators $\mathbf{M}$ *and* $\mathbf{K}$ *are self-adjoint and positive definite, and the operator* $\mathbf{G}$ *is skew-symmetric; that is,* $\mathbf{G}^* = -\mathbf{G}$. *Suppose that the eigenvalues* $\lambda_1, \ldots, \lambda_p$ *of (6.1.14) are distinct, the sets* $\{\mu_1, \ldots, \mu_p\}$ *and* $\{\lambda_1, \lambda_2, \ldots\}$ *are disjoint, and* $(\mathbf{b}, \mathbf{w}_j) \neq 0$ *for* $j = 1, \ldots, p$. *Let* f *and* g *be chosen as*

$$f = \sum_{j=1}^{p} \beta_j \lambda_j \mathbf{M}\mathbf{w}_j \quad \textit{and} \quad g = -\sum_{j=1}^{p} \beta_j \mathbf{K}\mathbf{w}_j\,, \tag{6.1.15}$$

where

$$\beta_j = \frac{1}{(\mathbf{b}, \lambda_j \mathbf{w}_j)} \frac{\prod_{i=1}^{p}(\overline{\mu}_i + \lambda_j)}{\prod_{\substack{i=1 \\ i\neq j}}^{p}(\lambda_j - \lambda_i)}\,. \tag{6.1.16}$$

Then, the closed-loop operator pencil

$$\mathbf{P}_c(\lambda)\phi = \lambda^2\mathbf{M} + \lambda(\mathbf{D} - (f,\phi)\mathbf{b}) + (\mathbf{K} - (g,\phi)\mathbf{b})$$

has the spectrum $\{\mu_1, \mu_2, \ldots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \ldots\}$.

Proof. The proof of Corollary 3.3 shows that every eigenvalue λ_j, $j = 1, 2, \ldots$ is purely imaginary and, moreover, $(\lambda_j, \mathbf{w}_j)$ is right eigenpair of (6.1.14) if and only if $(\lambda_j, \mathbf{w}_j)$ is also left eigenpair of (6.1.14); that is, $\mathbf{v}_j = \mathbf{w}_j$ for every j.

Since $\mathbf{K}$ is positive definite, Λ_1 is nonsingular. Define a vector $\beta = (\beta_1, \ldots, \beta_p)$ by

$$\beta = \Lambda_1^{-1}\Phi\,,$$

where Φ is given by (6.1.5). Since $\mathbf{M}$ and $\mathbf{K}$ are self-conjugate, we then obtain from (6.1.6)

$$f = \mathbf{f}_{11} = \sum_{j=1}^{p} \beta_j \lambda_j \mathbf{M}\mathbf{w}_j \quad \text{and} \quad g = \mathbf{f}_{21} = -\sum_{j=1}^{p} \beta_j \mathbf{K}\mathbf{w}_j\,,$$

proving (6.1.15).

We now show that the entries β_j are given by (6.1.5). Choose $\Gamma = (1, 1, \ldots, 1)$. Then (6.1.5) becomes

$$\beta\Lambda_1 Z_1 = (1, 1, \ldots, 1)\,. \tag{6.1.17}$$

Now, equating the like entries on both sides of (6.1.4), we obtain

$$z_{jk} \;=\; \frac{(\mathbf{w}_j, \mathbf{b})}{\lambda_j - \mu_k} \tag{6.1.18}$$

or, in the matrix notation,

$$Z_1 = \begin{pmatrix} (\mathbf{w}_1, \mathbf{b}) & & 0 \\ & \ddots & \\ 0 & & (\mathbf{w}_p, \mathbf{b}) \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1-\mu_1} & \cdots & \frac{1}{\lambda_1-\mu_p} \\ \vdots & \ddots & \vdots \\ \frac{1}{\lambda_p-\mu_1} & \cdots & \frac{1}{\lambda_p-\mu_p} \end{pmatrix}. \qquad (6.1.19)$$

Since the second factor of the right-hand side of (6.1.19) is a Cauchy matrix, using now the well known formula for the inverse of the Cauchy matrix, we obtain from (6.1.17) the required expressions for β_j's.

CHAPTER 7

NUMERICAL EXPERIMENTS

In this chapter, we present results of our numerical experiments on some real-life data with Algorithms 5.9, 5.12, and 6.4. Very satisfactory results have been obtained in each case.

7.1 Vibrations of a Rotating Turbine Axle

Following [54, Example 20] we consider a large and sparse symmetric definite quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ of order $n = 211$ modelling a rotating axle in a power plant, where masses are assumed to be symmetric with respect to the axle.

The matrix

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$$

is positive definite and the damping and stiffness matrices given by

$$D = (d_{ij}), \text{ where } d_{ij} = \begin{cases} -\gamma_i & , \quad i+1=j \\ \gamma_{i-1} + \delta_i + \gamma_i & , \quad i=j \\ -\gamma_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$K = (k_{ij}),\text{ where } k_{ij} = \begin{cases} -\kappa_i & , \quad i+1=j \\ \kappa_{i-1}+\kappa_i & , \quad i=j \\ -\kappa_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

with $\gamma_0 = \gamma_n = \kappa_0 = \kappa_n = 0$ are both symmetric tridiagonal.

Using the data provided in [54], the eigenvalues of the uncontrolled system are plotted in Figure 7.1.

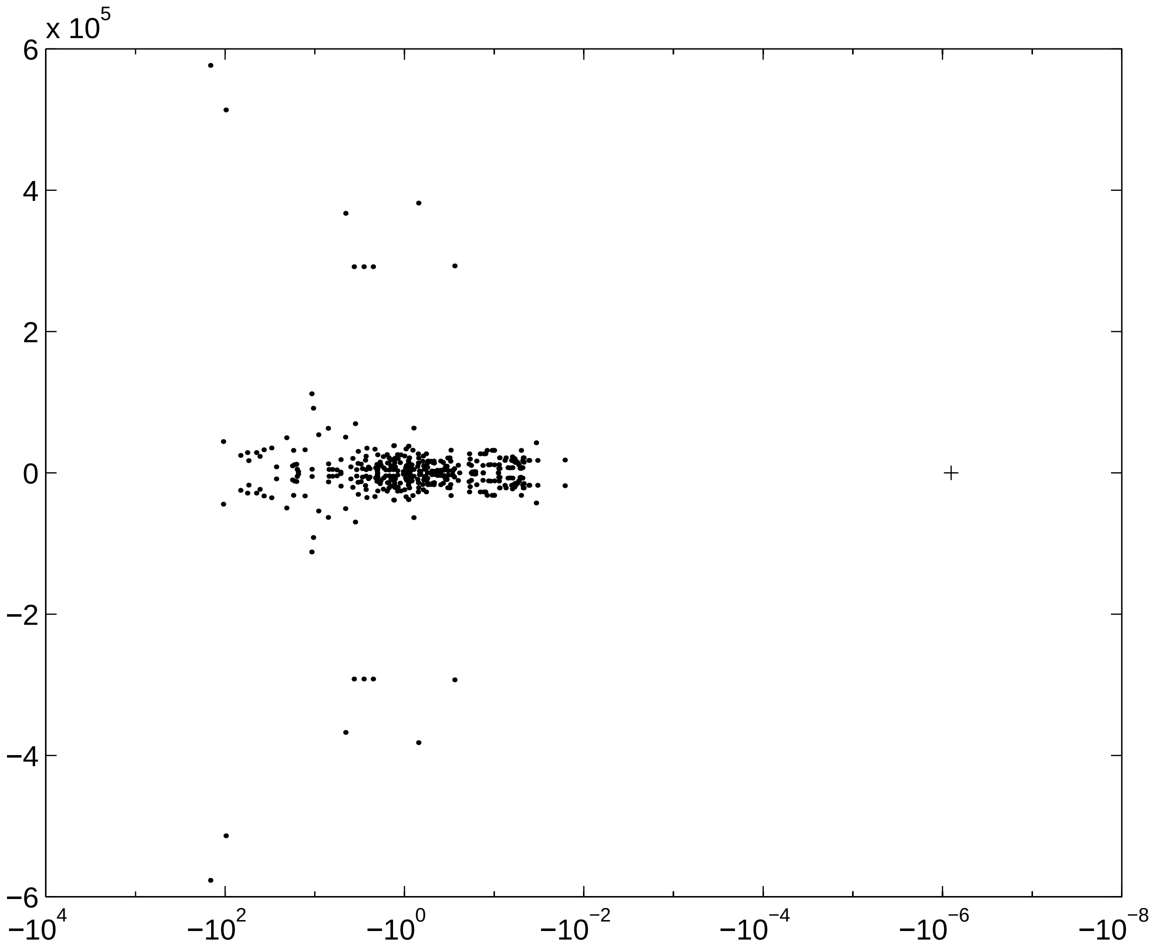

Figure 7.1: Open-Loop Eigenvalues of Rotating Turbine Axle; the Most Unstable Eigenvalue Is Marked by “+”.

It is clear that the decay rate of the vibrations of the axle is governed by its most unstable eigenvalue

$$\lambda_1 = -1.3734 \cdot 10^{-6}\,,$$

which is marked in Figure 7.1 by "+" sign, whereas the other eigenvalues have much better stability properties, namely

$$\mathrm{Re}\,\lambda_j \leq -0.016267\,,\ j = 2, 3, \ldots, 422.$$

7.1.1 Partial Eigenvalue Assignment for Rotating Turbine Axle

We choose a simple control matrix

$$B = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}^T$$

and apply our Algorithm 5.9 to assign λ_1 to $\mu_1 = -0.016$ so that the decay rate is improved by the factor of 10^4.

With random choice of the matrix $\Gamma = (-0.51454, -0.85747)^T$ the computed feedback matrices F_1 and F_2 are such that μ_1 was assigned correctly while the 2-norm of the difference between the other eigenvalues of the open-loop pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ and the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF_1) + (K - BF_2)$ is about $1.7 \cdot 10^{-6}$ (MATLAB was used to compute the eigenvalues). The 2×422 matrices F_1 and F_2 are not reproduced here because of the space limitation; however, we note that $||F_1||_2 < 116$ and $||F_2||_2 < 22$. Furthermore,

$$\frac{||F_1||_2}{||C||_2} < 0.57 \text{ and } \frac{||F_2||_2}{||K||_2} < 1.5 \cdot 10^{-11}\,,$$

which improves the result of [19] by reducing the control forces required to suppress the vibrations of the rotating turbine axle nearly 10^3-fold.

7.1.2 Partial Eigenstructure Assignment for Rotating Turbine Axle

Since the largest contribution to shape of the transient response of the vibrating system is generated by the eigenvector that corresponds to the most unstable eigenvalue of the system, we use Algorithm 5.12 to assign λ_1 to $\mu_1 = -0.016$ and, simultaneously, to assign the eigenvector corresponding to λ_1 to the vector

$$x_{c1} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T \,,\; n = 211\,.$$

Algorithm 5.12 produces the 211×1 control matrix B with components shown in Figure 7.2 and with $||B||_2 < 2$ and the 1×211 feedback matrices F_1 and F_2 with $||F_1||_2 < 7.2$ and $||F_2||_2 < 1.4$, respectively, such that the prescribed eigenvalue and eigenvector are assigned correctly. Moreover,

$$\frac{||BF_1||_2}{||C||_2} < 0.07 \text{ and } \frac{||BF_2||_2}{||K||_2} < 1.8 \cdot 10^{-12}\,.$$

This shows that control forces required to suppress vibrations assigning the same eigenvalue are 10 times less than those required by eigenvalue assignment with a priori control matrix B. To achieve this, however, we need more sophisticated actuators than those needed to implement the simple control force used in eigenvalue assignment.

The computed matrices B, F_1, and F_2 are such that the 2-norm of the differences between the remaining eigenvalues of the open-loop pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ and the corresponding ones of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF_1) + (K - BF_2)$ is about $2.2 \cdot 10^{-6}$ (MATLAB was used to compute the eigenvalues).

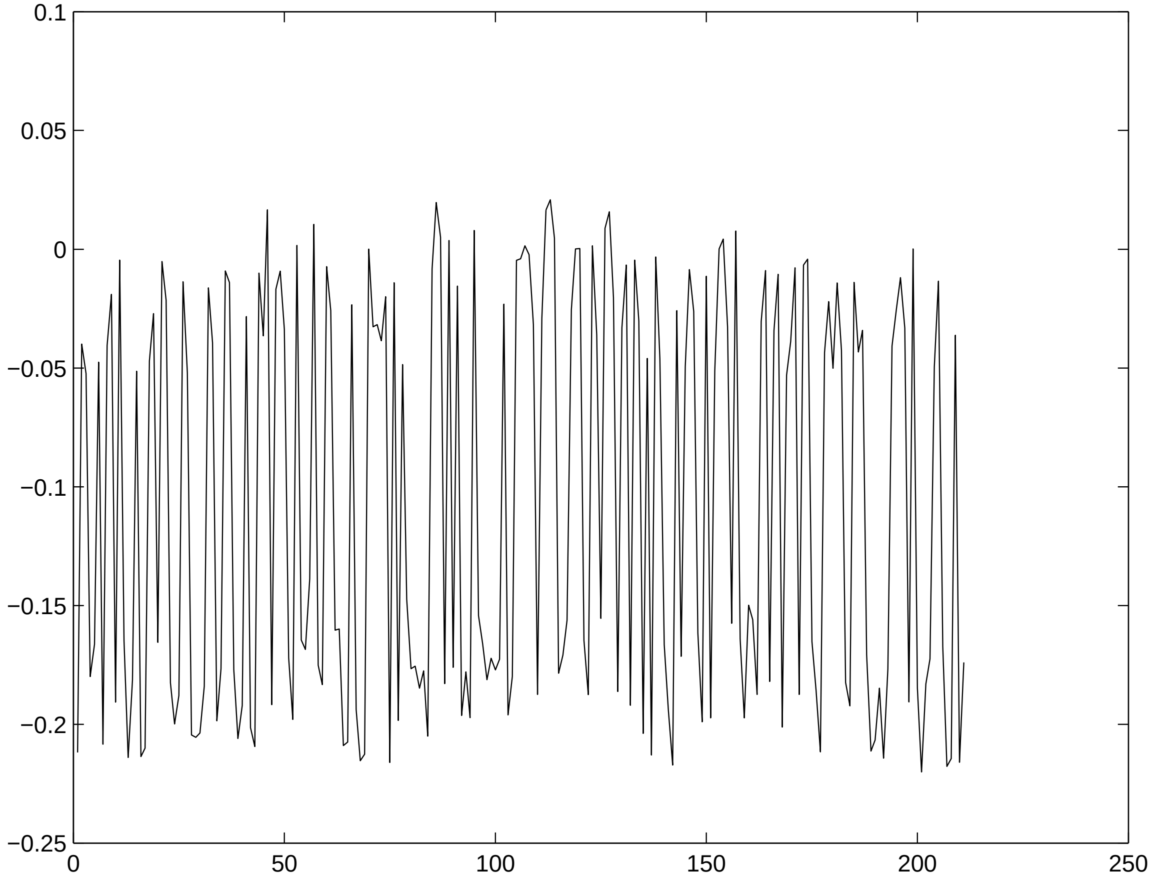

Figure 7.2: The Components of the Control Matrix B.

7.2 Vibrations of a Uniform Moving String

Following [18, 22], we consider a gyroscopic operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M}+\lambda\mathbf{G}+\mathbf{K}$ modelling the small oscillations of a uniform string travelling with constant velocity $\gamma < c$ over two fixed supports at $x = 0$ and $x = L$. With $L = 1$, $\gamma = 1/2$, and $c = 1$, the operators $\mathbf{M}$, $\mathbf{G}$, and $\mathbf{K}$, defined in Section 2.1, are given by

$$\mathbf{M}v = v\,,\ \mathbf{G}v = \frac{\partial v}{\partial x}\,,\ \mathbf{K}v = \frac{3}{4}\frac{\partial^2 v}{\partial x^2}\,,$$

where $v(0) = v(1) = 0$, with the scalar product

$$(v, w) \quad = \quad \int_0^1 \overline{v(x)} w(x) \; dx \, .$$

As shown in Section 2.1, the operators $\mathbf{M}$, $\mathbf{G}$, and $\mathbf{K}$ have the following properties:

$$\mathbf{M}^* = \mathbf{M} \, , \; \mathbf{G}^* = -\mathbf{G} \text{ and } \mathbf{K}^* = \mathbf{K} \, .$$

Recall, that left eigenfunctions of operator pencil $\mathbf{P}(\lambda)$ are defined by (3.4.50) as the eigenfunctions of the adjoint pencil

$$\mathbf{P}(\lambda)^* = \lambda^2 \mathbf{M}^* + \lambda \mathbf{G}^* + \mathbf{K}^* \, .$$

Thus the eigenvalues of $\mathbf{P}(\lambda)$ are

$$\lambda_k \quad = \quad \frac{3}{4} \pi k i \, , \; k = \pm 1, \pm 2, \ldots \qquad (7.2.1)$$

and their corresponding left eigenfunctions are

$$\mathbf{v}_{ck}(x) \quad = \quad e^{\frac{3}{2} \pi k i x} - e^{-\frac{1}{2} \pi k i x}, \text{ where } 0 \le x \le 1 \, . \qquad (7.2.2)$$

Algorithm 6.4 is used to assign λ_1 to $\mu_1 = -1 + i$ and $\overline{\lambda}_1$ to $\overline{\mu}_1$, choosing the two control functions

$$\mathbf{b}_1(x) = 1 \text{ and } \mathbf{b}_2(x) = \sin(\pi x), \text{ where } 0 \le x \le 1 \, .$$

The step-wise results of our implementations are given in the following:

Step 1. We form

$$\mathbf{\Lambda}_1 = \operatorname{diag}\left(\frac{3}{4}\pi i, -\frac{3}{4}\pi i\right)\ ,\ \mathbf{V}_1 = \left(e^{\frac{3}{2}\pi i x} - e^{-\frac{1}{2}\pi i x}, e^{-\frac{3}{2}\pi i x} - e^{\frac{1}{2}\pi i x}\right)$$

and

$$\mathbf{\Lambda}_{c1} = \operatorname{diag}(-1+i, -1-i)\,.$$

Step 2. We choose arbitrary

$$\Gamma = \begin{pmatrix} -0.20575 + 0.8342i & -0.20575 - 0.8342i \\ 0.22626 + 0.45888i & 0.22626 + 0.45888i \end{pmatrix}.$$

Step 3. Solving the Sylvester equation for Z_1,

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (\mathbf{v}_1, \mathbf{b}_1) & (\mathbf{v}_1, \mathbf{b}_2) \\ (\mathbf{v}_2, \mathbf{b}_1) & (\mathbf{v}_2, \mathbf{b}_2) \end{pmatrix} \Gamma\,,$$

we obtain

$$Z_1 = \begin{pmatrix} 0.18844 + 0.36623i & -0.14535 - 0.84357i \\ -0.14535 + 0.84357i & 0.18844 - 0.36623i \end{pmatrix}.$$

Step 4. Solving $\Phi Z_1 = \Gamma$ for Φ_1, we obtain

$$\Phi = \begin{pmatrix} 0.82911 - 0.50588i & 0.82911 + 0.50588i \\ 0.25486 + 0.45099i & 0.25486 - 0.45099i \end{pmatrix}.$$

Step 5. The velocity feedback functions $\mathbf{f}_{11}$ and $\mathbf{f}_{12}$, plotted in Figure 7.3, are given by

$$\mathbf{f}_{11} \quad = \quad -4.0471\cos^2\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) - 6.6329\cos\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right)$$

$$
\begin{aligned}
\mathbf{f}_{12} \;=\;& 3.60796\cos^6\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) - 2.03892\cos^5\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right) + \\
& 7.21593\cos^4\left(\frac{\pi x}{2}\right)\sin^3\left(\frac{\pi x}{2}\right) - 4.07785\cos^3\left(\frac{\pi x}{2}\right)\sin^4\left(\frac{\pi x}{2}\right) + \\
& 3.60796\cos^2\left(\frac{\pi x}{2}\right)\sin^5\left(\frac{\pi x}{2}\right) - 2.03892\cos\left(\frac{\pi x}{2}\right)\sin^6\left(\frac{\pi x}{2}\right)
\end{aligned}
$$

and the position feedback functions $\mathbf{f}_{21}$ and $\mathbf{f}_{22}$, plotted in Figure 7.4, are given by

$$
\begin{aligned}
\mathbf{f}_{21} \;=\;& 6.3571\cos^3\left(\frac{\pi x}{2}\right) + 36.466\cos^2\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) - \\
& 22.25\cos\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right) - 10.418\sin^3\left(\frac{\pi x}{2}\right) \\
\mathbf{f}_{22} \;=\;& -5.6673\cos^3\left(\frac{\pi x}{2}\right) + 11.209\cos^2\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right) + \\
& 19.835\cos\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right) - 3.2027\sin^3\left(\frac{\pi x}{2}\right) .
\end{aligned}
$$

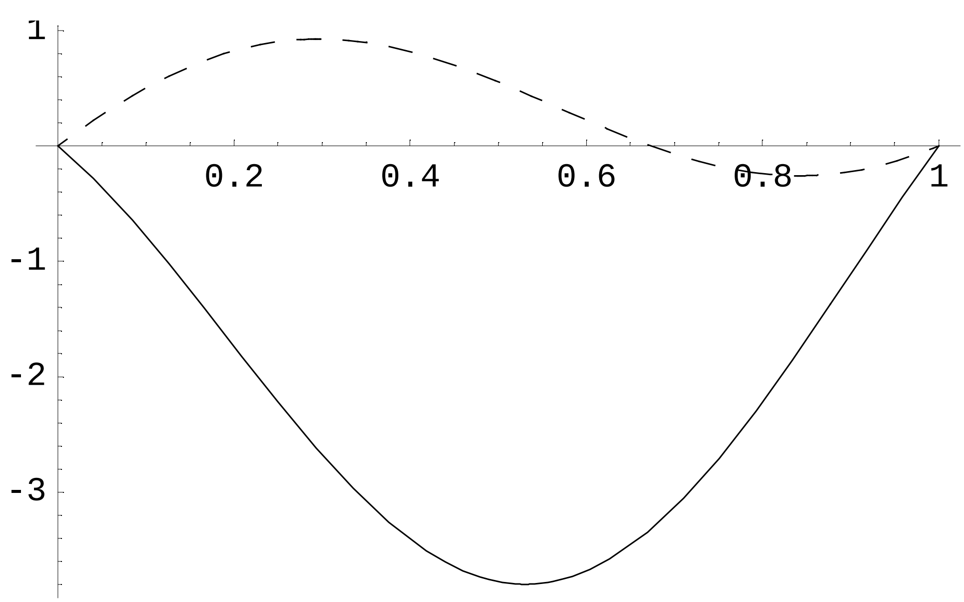

Figure 7.3: The Velocity Feedback Functions $\mathbf{f}_{11}$ and $\mathbf{f}_{12}$.

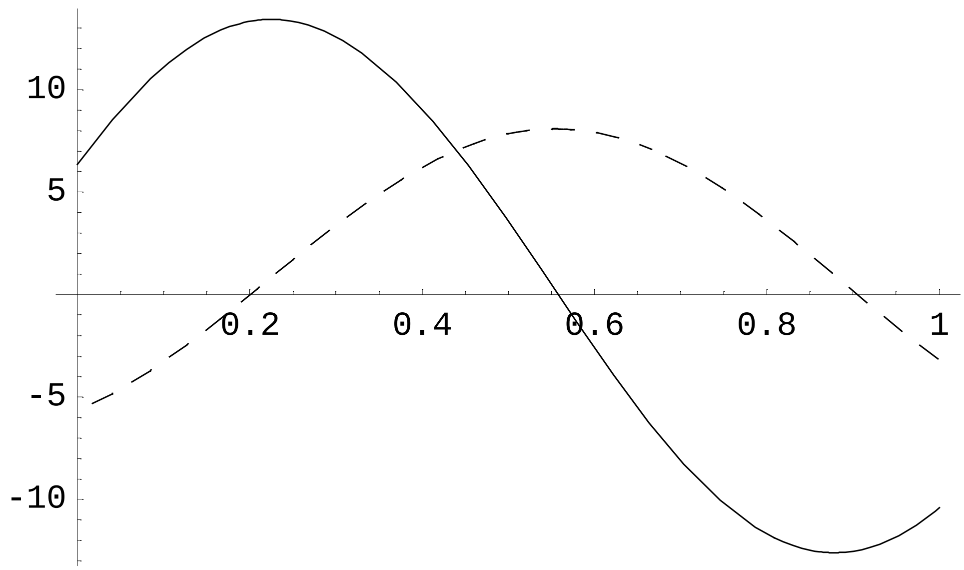

Figure 7.4: The Position Feedback Functions $\mathbf{f}_{21}$ and $\mathbf{f}_{22}$.

The closed-loop operator pencil

$$\mathbf{P}_c(\lambda)(v) \quad = \quad \lambda^2 \mathbf{M}v + \lambda \mathbf{G}v + \mathbf{K}v - \sum_{k=1}^{2} \left((\mathbf{f}_{1k}, \lambda v) + (\mathbf{f}_{2k}, v) \right) \mathbf{b}_k$$

has the eigenvalues μ_1 and $\overline{\mu}_1$ with eigenfunctions given by

$$\begin{aligned} \mathbf{w}_{c1} \quad = \quad & (0.4171 + 0.10287i) - (0.23671 + 0.20962i)e^{-2(1+i)x} - \\ & (0.19416 - 0.07776i)e^{\frac{2}{3}(1+i)x} - (0.0088786 - 0.037267i)e^{-\pi i x} + \\ & (0.022656 - 0.0082765i)e^{\pi i x} \end{aligned}$$

and $\overline{\mathbf{w}_{c1}(x)}$, respectively. Furthermore, the eigenvalues λ_k, $k = \pm 2, \pm 3, \ldots$, of the open-loop pencil $\mathbf{P}(\lambda) = \lambda^2 \mathbf{M} + \lambda \mathbf{G} + \mathbf{K}$ remain unchanged.

CHAPTER 8

CONCLUSIONS

In this chapter, we present some concluding remarks. In Section 8.1, we summarize our contributions and state their importance. In Section 8.2, we state some future research problems. Finally, in Section 8.3, we remark about the academic and industrial impacts of the results obtained in this dissertation.

8.1 Contributions

In this dissertation, we have developed a novel approach, named "partial modal approach," for two important feedback control problems, namely, the partial eigenvalue assignment and partial eigenstructure assignment problems for control systems modelled by finite-dimensional matrix second-order systems.

The approach has also been extended to the partial eigenvalue assignment in distributed-parameter systems, which are the natural mathematical models of vibrating structures and of which the matrix second-order systems are just the discretized versions, obtained by using finite-element technique.

There are several distinctive features of this approach.

First, the problems are solved in their given settings; that is, if the problem is given for a matrix second-order model, it is solved without reformulation to a first-order system. If the model is a distributed-parameter system, the problem is solved without discretization to a second-order system. The advantages are that some of the nice inherited properties such as symmetry, definiteness, sparsity, etc., can be exploited in computations of the feedback matrices and operators.

Second, the approach requires knowledge of only a small number of eigenvalues and eigenvectors of the associated open-loop quadratic pencil, making the proposed methods implementable in practice, even for very large and sparse systems using the state-of-the-art techniques of matrix computations. This aspect is especially remarkable in the distributed-parameter systems case because here an infinite-dimensional operator problem is solved using a small finite number of eigenvalues and eigenvectors.

Third, mathematical results are proven in each case, guaranteeing no spill-over; that is, the eigenvalues and eigenvectors that are to remain invariant do not change by application of feedback.

Besides computational algorithms, the dissertation contains some new theoretical results on the existence and uniqueness of the solutions of the partial eigenvalue and eigenstructure assignment problems, both in matrix first-order and second-order systems. Also, new results on the Rayleigh quotient and the orthogonality relations between the eigenvectors of a quadratic pencil are derived.

In addition to their role in algorithmic development, these results are of independent interests and are important contributions to the field of linear algebra and applied mathematics.

8.2 Future Research

Based on the theoretical investigation and numerical experiments, we propose the following problems for future research.

- **Robust Eigenvalue Assignment.**

  It is well known that one of the main contributing factors to the sensitivity of the closed-loop eigenvalues is the condition number of the eigenvector matrix.

In our parametric solution to these problems, there are opportunities to choose the matrix of parameters Γ to make the eigenvector matrix as well conditioned as possible. We believe that the condition number of the matrix Z_1 appearing in all the algorithms will play a crucial role here. Our numerical experiments on the sensitivity of the closed-loop eigenvalues under perturbations of the system and feedback matrices, as the condition number of the matrix Z_1 varies, confirm our belief. Development of a mathematical theory is in order.

- **Partial and Modal Approach for Eigenstructure Assignment in Distributed Parameter Systems.**

  The partial and modal approach proposed for the eigenvalue and eigenstructure assignment in matrix second-order systems and for the eigenvalue assignment in distributed-parameter systems should, naturally, be extended to the partial eigenstructure assignment in distributed-parameter systems. Preliminary work to this effect has been done and details need to be worked out.

- **Finite-Element Model Updating.**

  The *finite-element model updating problem* in vibration analysis and design is the problem of updating a finite-element generated model such that a few eigenvalues and corresponding eigenvectors are replaced by the given set of measured eigenvalues and eigenvectors, keeping the remaining eigenvalues and eigenvectors unchanged, and the updated model is symmetric.

  Our proposed method for the eigenstructure assignment problem for the quadratic matrix pencil solves the finite-element model updating problem, except that the symmetry is not preserved. The future research should be directed toward investigating how to bring back symmetry of the updated model after application of the feedback.

8.3 Impact

The results obtained in this dissertation are expected to set a new direction of research on solutions of control problems in matrix second-order and distributed-parameter systems. These results are also likely to impact a variety of industries including automotive, aerospace, power plants, etc.

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