RECENT ADVANCES ON COMPUTATIONAL METHODS FOR STRUCTURED INVERSE QUADRATIC EIGENVALUE PROBLEMS

by
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Two Inverse Quadratic Eigenvalue Problems

I. Quadratic Partial Eigenvalue Assignment Problem **(QPEVAP)**

Controlling Dangerous Vibrations in Structures

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QPEVAP

II. Finite Element Model Updating Problem **(FEMUP)**.

Updating Theoretical FEM Using Measured Data from Real-Life Structure

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FEMUP ≡ **Structure preserving QPESAP**

The Quadratic Eigenvalue Problem:

$$(\lambda^2 M + \lambda D + K)x = 0$$

- $2n$ eigenvalues and $2n$ corresponding eigenvectors.
- The eigenvalues are the roots of the quadratic pencil $\det(\lambda^2 M + \lambda D + K) = 0$.

- **Quadratic Matrix Pencil**

$$P(\lambda) = \lambda^2 M + \lambda D + K$$

Generalization of Standard Eigenvalue Problem

$$Ax = \lambda x$$

and
the **Generalized Eigenvalue Problem**

$$Ax = \lambda Bx.$$

Approach I

- Reduction to a Standard $2n \times 2n$ Eigenvalue Problem

$$Au = \lambda u$$

where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

$$u = \begin{pmatrix} x \\ \lambda x \end{pmatrix}.$$

(Assuming that M is nonsingular)

- The eigenvalues are the same
- The eigenvectors are extracted from the eigenvectors u.

Numerical Difficulties

- M **is ill-conditioned**.
- Special structural properties: *definiteness, sparsity, bandness*, etc. **destroyed**.

- **Reduction to a Generalized Eigenvalue Problem:**

Symmetric Generalized Eigenvalue Problem

$$Bz = \lambda C z$$

$$\bullet B = \begin{pmatrix} D & K \\ K & 0 \end{pmatrix}$$

$$\bullet C = \begin{pmatrix} -M & 0 \\ 0 & K \end{pmatrix}$$

$$\bullet z = \begin{pmatrix} \lambda x \\ x \end{pmatrix}$$

Numerical Difficulties

The pencil $Bz = \lambda Cz$ is symmetric, but in general **indefinite**, even though M, K, and D are symmetric positive definite.

Remark: The QEP is **nonlinear eigenvalue problem** - *difficult to solve.*

State - of the - Art Methods.

- A **Look-ahead Lanczos Algorithm** of Parlett and Chen (1980) (only a few extremal eigenvalues).
- The **Jacobi-Davidson Method** (Projection Method).

Only a few extremal eigenvalues and eigenvectors computed.

Applications of the QEP.

- Vibration Analysis of Structural Mechanical and Acoustic Systems
- Electrical Circuit Simulation
- Fluids Mechanics
- Modeling Microelectronic
- Finite-Element Model Updating in Aerospace and Automobile Industries.

Quadratic Inverse Eigenvalue Problems.

- Certain inverse eigenvalue problems for the quadratic pencil arising in practical applications can be handled with a small number of eigenvalues and eigenvectors, if done properly.

Examples of Resonance

Dangerous vibrations such as **resonance** are caused by a few bad eigenvalues.

Classical Examples of Resonance:

- The Fall of the Tacoma Bridge
- The Fall of the Broughton Bridge in England
- Wobbling of the Millennium Bridge over the River Thames in London, England

  (www.arup.com/Millenniumbridge)

Phenomenon of Resonance

- **The Discretized Finite Element Model**

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0.$$

- **The Associated Quadratic Matrix Eigenvalue Problem:**

$$(\lambda^2 M + \lambda D + K)x = 0.$$

- The dynamics are governed by

  Natural Frequencies $\longrightarrow$ Eigenvalues of the QEP.

  Mode Shapes $\equiv$ Eigenvectors of the QEP.

Response of a Structure due to Harmonic Input

$$j = \sqrt{(-1)}.$$

- $f(t) =$ External Force $= f_o\, e^{j\omega t}$
- Oscillatory Solution $x(t) = x(t)e^{j\omega t}$
- $(K + j\omega D - \omega^2 M)xe^{j\omega t} = f_o e^{j\omega t}$
- $x = (K + j\omega D - \omega^2 M)^{-1} f_o$ **(Response)**.

As

$$j\omega \rightarrow \lambda_j$$

$||P(j\omega)^{-1}||$ increases without bound.

- Resonance is caused by closed proximity of an external frequency to that of a natural frequency.

How to Avoid Resonance?

- Feedback Control can be used

Idea: Replace {computed Unwanted eigenvalues}
$\longrightarrow$ {suitably chosen ones}

and

Leave the remaining large number unchanged.

(No spill-over)

Feedback Control in Second-order Model

A possible Remedy: Apply a suitable control force to the structure. Use the technique of **feedback control**.

- **Matrix Second-order Model with Control**

$$\boxed{M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t)}$$

B - Control Matrix
$u(t)$ - Control Vector

- **Second-order Feedback Closed-loop System**

Choose $u(t) = F_1\dot{x}(t) + F_2x(t)$.

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = B(F_1\dot{x}(t) + F_2x(t))$$

$$\boxed{M\ddot{x}(t) + (D - BF_1)\dot{x}(t) + (K - BF_2)x(t) = 0.}$$

The associated matrix quadratic pencil:

- $P_c(\lambda) = \boxed{\lambda^2 M + \lambda(D - BF_1) + (K - BF_2) = 0.}$

This pencil is called the **closed-loop pencil**.

Notations

- **The spectrum of the quadratic pencil:**

$$\Omega(P(\lambda)) = \{\lambda_1, ..., \lambda_p; \lambda_{p+1}, ..., \lambda_{2n}\}$$

- **The right eigenvectors of the:**

$$\{x_1, ..., x_p;\ x_{p+1}, ..., x_{2n}\}$$

- **The left eigenvectors of the pencil:**

$$\{y_1, \ldots, y_p; y_{p+1} \ldots, y_{2n}\}.$$

Quadratic Partial Eigenvalue Assignment Problem (QPEVAP)

Given

- The system matrices $M, K, D, \in \mathbb{R}^{n\times n}(M = M^T > 0, \quad K = K^T \geq 0$ and $D = D^T)$.
- A control matrix $B \in \mathbb{R}^{n\times m}$

Find the Feedback Matrices F_1 and F_2 such that

$$\Omega(P_c(\lambda)) = \{\mu_1, \ldots, \mu_p;\ \lambda_{p+1}, \ldots, \lambda_{2n}\}.$$

- {Unwanted Eigenvalues} $\longrightarrow$ {User's Chosen Eigenvalues}
- {Good Eigenvalues} = {**Remain Unchanged**}

Stabilizing a Second-order System

(A Special Case)

- Solution of the QPEVA problem can be used to stabilize a matrix second-order system by feedback.

Two Standard Approaches for Control

- Solution via transformation to a **first-order State-Space Form**
- **Independent Modal Space Control** (IMSC) Approach.

  Both these approaches have severe computational difficulties and engineering limitations.

Approach I

Standard First-order Reduction

Recall the second-order feedback control system

$$M\ddot{x}(t) + (D - BF_1)\dot{x}(t) + (K - BF_2)x(t) = 0.$$

- Reduction to Standard First-order State-space Form:

$$\dot{q}(t) = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} q(t) + \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix} u(t)$$

Opportunities

- Many numerically excellent methods can be used (**Numerical Methods for Linear Control Systems Design and Analysis,** by B.N. Datta, Elsevier Academic Press, 2003)

Difficulties

- Ill-conditioned matrix inversion might be necessary.
- All important structures such as *sparsity, definiteness and bandness* etc. are lost.
- Problem size becomes double.

Non-standard first-order reduction:

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \dot{z}(t) = \begin{pmatrix} 0 & -K \\ -K & -D \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ B \end{pmatrix} u(t)$$

or

$E\dot{z}(t) = Az(t) + \hat{B}u(t)$ **(Descriptor System)**

- Numerical methods for descriptor systems not well-developed (E could be *singular* or *very ill-conditioned*)
- Symmetry preserved, but *not Positive Definiteness, Spartity and other Properties.*

Approach II
Independent Space Control (IMSC) Approach.

(For Open-loop Decoupling)

- Requires complete knowledge of the spectrum and eigenvectors of the open-loop pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K.$$

Impractical for large and sparse problems

(For closed-loop Decoupling)

$$\boxed{\begin{aligned} BKM^{-1}D &= DM^{-1}BK \\ BKM^{-1}K &= KM^{-1}BK \end{aligned}}$$

- Stringent requirements need to be satisfied on actuators and sensors which are impossible to satisfy in practice.

Ref: Vibration with Control, Measurement, and Stability by D. Inman, Prentice Hall, 1989.

Challenges

- Use a **small number of eigenvalues and eigenvectors** that can be computed or measured.
- **No transformation** to a first-order system.
- **No reduction of the order** of the model or the order of the controllers.
- **Mathematical guarantee** needed for the **no spillover property**.

The Current Engineering Practice and Drawbacks

- Compute and control the first few frequencies and mode shapes (eigenvalues and eigenvectors).
- Hope that the large number of remaining eigenvalues and eigenvectors do not chan ge or do not spill-over to dangerous regions.
- **Unfortunately, the spill-over almost always occurs**.
- **No mathematical basis**

Recent Direct and Partial-Modal Approach for Feedback Control

(Collaborative work with **Eric Chu, Sylvan Elhay, Yitshak Ram, Daniil Sarkissian, W.W. Lin, J.N. Wang, and others**)

- **Direct** - No transformation required.
- **Partial-Modal** - Only knowledge of a small number of eigenvalues and eigenvectors needed for implementation.
- Extension to the **Robust Partial Eigenvalue Assignment.** (Sensitivity minimization by minimization of the *eigenvector condition number and feedback normly*)

A New Approach for the Quadratic Partial Eigenvalue Assignment Problem

- Two-part solution

  Part I. **No spill-over part** (with a parametric matrix).

  Part II. **Partial Eigenvalue Assignment Part**. (with a special choice of the parametric matrix)

Notations

Define $\Lambda_1 = \text{ diag } (\lambda_1, \ldots, \lambda_p)$

$Y_1 = (y_1, y_2, \ldots, y_p)$

$\Lambda_{cl} = \text{ diag } (\mu_1, \ldots, \mu_p)$.

Solution of Part I

Theorem on No Spill-over

- Choose any **arbitrary parametric matrix** Φ
- Define

$$F_1 = \Phi Y_1^H M$$

and

$$F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H D)$$

Then

$$\Omega(\lambda^2 M + \lambda(D - BF_1) + (K - BF_2)) = \{** \cdots *, \lambda_{p+1}, \ldots, \lambda_{2n}\}.$$

No Change.

Note: Only small number of eigenvalues needed for constructing F_1 and F_2.

New Orthogonality Results on the Eigenvectors of the Quadratic Matrix Pencil

Assume

$$\{\lambda_1, \cdots, \lambda_p\} \cap \{\lambda_{p+1}, \cdots, \lambda_{2n}\}) = \phi.$$

Partition $\Lambda = \text{ diag } (\Lambda_1, \Lambda_2)$

$$\begin{aligned} X &= (X_1, X_2) \\ Y &= (Y_1, Y_2) \end{aligned}$$

Then

- $\Lambda_1 Y_1^H M X_2 \Lambda_2 - Y_1^H K X_2 = 0$

  and

- $\Lambda_1 Y_1^H M X_2 + Y_1^H M X_2 \Lambda_2 + Y_1^H D X_2 = 0.$

Generalization of Orthogonality Results of SEVP and SDGEVP

- $X^T A X = \text{Diagonal}$ (Symmetric EVP)

- $$\left.\begin{aligned} X^T A X &= \text{Diagonal} \\ X^T B X &= I \end{aligned}\right] \text{Symmetric Definite GEVP}$$

Solution of Part II (How to Choose Φ?)

Theorem on Partial Eigenvalue Assignment

- Solve the $p \times p$ Sylvester Equation.

$$\Lambda_1 Z_1 - Z_1 \Lambda_{cl} = Y_1^H BF$$

- Solve the $p \times p$ Linear system $\Phi Z_1 = \Gamma$ by choosing Γ arbitrary.

- **Result**:

$$\Omega(\lambda^2 M + \lambda(D - BF_1) + (K - BF_2)) =$$
$$\{\underbrace{\mu_1, \ldots, \mu_p;}_{\textbf{Desiresed EVS}} \qquad \underbrace{\lambda_{p+1}, \ldots, \lambda_{2n}}_{\textbf{No Change}}\}.$$

An Algorithm for QPEVAP

Step 1. Form

- $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$
- $Y_1 = (y_1, \ldots, y_p)$
- $\Lambda_{c1} = \text{diag}(\mu_1, \ldots, \mu_p)$.

Step 2. Choose arbitrary $m \times 1$ vectors $\gamma_1, \ldots, \gamma_p$ in such a way that $\overline{\mu_j} = \mu_k$ implies $\overline{\gamma_j} = \gamma_k$ and form $\Gamma = (\gamma_1, \ldots, \gamma_p)$.

Step 3. Find the unique solution Z_1 of the $p \times p$ Sylvester equation

$$\Lambda_1 Z_1 - Z_1 \Lambda_{cl} = Y_1^H B \Gamma.$$

If Z_1 is ill-conditioned, then return to Step 2 and select different $\gamma_1, \ldots, \gamma_p$.

Step 4. Solve $\Phi Z_1 = \Gamma$ for Φ.

Step 5. Form $F_1 = \Phi Y_1^H$ and $F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H D)$.

- Standard Numerical Methods for Solving Sylvester and Lyapunov Equations

- **Numerical Methods for Linear Control Systems (Chapter 8)**.

Computing Resources and Requirements for Implementations

- A small number of eigenvalues and eigenvectors
- Solution of a small Sylvester equation
- Solution of a small linear algebraic system

Practical and Computational Features

- Applicable to even very large real-life structures
- No transformation or model reduction
- Suitable for high-performance computing
  (Rich in BLAS-3 Computations.)
- Sparsity, bandness, symmetry, etc. can be exploited
- Mathematical guarantee of no spill-over
- Extension to more general problem of both partial **eigenvalue** and **eigenvector assignment (QPESA)**
- Generalization to the Partial Eigenvalue Assignment in **DPS**. **(Infinite Dimensions).**

Quadratic Partial Eigenstructure Assignment Problem (QPEASP)

Given

- The system matrices $M, K, D, \in \mathbb{R}^{n\times n}(M = M^T > 0, \quad K = K^T \geq 0$ and $D = D^T)$.
- A set of computed unwanted eigenvalues $\{\lambda_1, ..., \lambda_p\}$.
- A set of user's chosen eigenvalues $\{\mu_1, ..., \mu_p\}$.
- A set of user's chosen eigenvectors $\{x_{c1}, \ldots, x_{cp}\}$

Find the Feedback Matrices F_1 and F_2 and a control matrix B such that

$$\Omega(P_c(\lambda)) = \{\mu_1, \ldots, \mu_p;\ \lambda_{p+1}, \ldots, \lambda_{2n}\}.$$

The Eigenvectors of $p_c(\lambda) = \{x_{cl}, \ldots, x_{cp}; x_{p+1}, x_{2n}\}$.

$\{$Unwanted Eigenvalues and Eigenvectors$\}$ $\longrightarrow$ $\{$User's Chosen Eigenvalues and Eigenvectors$\}$

$\{$Remaining Eigenvalues and Eigenvectors $\longrightarrow$ No Change.$\}$

An Algorithm for QPESA

Step 1. Form $\Lambda_1 = \text{ diag}(\lambda_1, \ldots, \lambda_p)$,

$$Y_1 = (y_1, \ldots, y_p),$$

$$\Lambda_{c1} = \text{ diag}(\mu_1, \ldots, \mu_p), \text{ and } (X_{cl}, \ldots, x_{cp}).$$

Step 2. Form the matrix

$$Z_1 = \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1}.$$

Stop if Z_1 is singular and conclude that the eigenstructure assignment with the given sets of eigenvalues and eigenvectors is not possible.

Step 3. Form the matrix T_c such that $T_c \Lambda_{c1} T_c^H$ is a real matrix.

Step 4. Form

$$B = (M X_{cl} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}) T_c^H,$$

$$F_1 = T_c Z_1^{-1} Y_1^H M, \text{ and}$$

$$F_2 = T_c Z_1^{-1} (\Lambda_1 Y_1^H M + Y_1^H C)$$

by solving the appropriate linear systems.

- There also exists a parametric Algorithm (as that of QPEVA)

  (Ph.D Thesis by **Daniil Sarkissian**, Northern Illinois University, 2001).

Natural Mathematical Model

Distributed Parameter Systems

$\downarrow$

$\downarrow$

FEM

$\downarrow$

$\downarrow$

Discretized Finite Element Model

System of Second-order ODE.

• Distributed Parameter Systems Model (DPS)

Distributed Parameter Systems:

$$M(x)\frac{\partial^2 \nu(t,x)}{\partial t^2} + C(x)\frac{\partial \nu(t,x)}{\partial t} + K(x)\nu(t,x) = 0.$$

M, C, and K are **differential operators** in the x-domain (spatial domain) of the displacement function $\nu(t,x)$.

$\nu(t,x)$ belongs to some Hilbert space.

$M =$ **Mass operator** (Self Adjoint)
$K =$ **Stiffness operator** (Self Adjoint)
$C = D + G$
$D =$ **Damping operator**
$G =$ **Gyroscopic operator** (Skew Symmetric)

DPS problems are **infinite dimensional**.

Two Additional Fundamental Challenges

- Use finite dimensional control and computational techniques
- Guarantee the invariance of the finite spectrum mathematically.

Mathematical Statement of the PEVA in DPS

Given

- The operators M, C, and K, of the DPS
- A self conjugate set of numbers $\{\mu_1, \ldots, \mu_p\}$
- Suitable control functions $b_1, \ldots, b_m$.

Find Real Feedback Functions $f_{11}, \ldots, f_{1m}$ and $f_{21}, \ldots, f_{2m}$ such that

$$\begin{aligned} \Omega(P_c(\lambda)\phi) &= \lambda^2 M\phi + \lambda(C\phi - \sum_{k=1}^{m}(f_{1k}, \phi)_k) \\ &+(K\phi - \sum_{k=1}^{m}(f_{2k}, \phi)_k) \end{aligned} \tag{1}$$

is the set $S = \{\mu_1, \cdots, \mu_p;\ \lambda_{p+1}, \lambda_{p+2}, \cdots\}$.

III. Partial Eigenvalue Assignment (PEVA) in Distributed Parameter Systems

Reassign a small part of the infinite open-loop spectrum of the operator pencil $P(\lambda) = \lambda^2 M + \lambda C + K$, by using feedback such that

i. the set is replaced by a suitable chosen set

ii. the remaining infinitely many eigenvalues do not change

$$\{\lambda_1, \ldots, \lambda_p\} \Longrightarrow \{\mu_1, \ldots, \mu_p\}$$

$$\{\lambda_{p+1}, \ldots\} \Longrightarrow \{\lambda_{p+1}, \ldots\}$$

No Change

Theorem **(Parametric Solution to the Partial Eigenvalue Assignment Problem for a Quadratic Operator Pencil).**

Part (i) (No-spill-over Part).
Choose Φ_{kj} arbitrarily and define

$$f_{1k} = \sum_{j=1}^{p} \bar{\Phi}_{kj} M^* v_j$$
$$f_{2k} = \sum_{j=1}^{p} \bar{\Phi}_{kj} (\bar{\lambda}_j M^* v_j + C^* v_j),$$

Result:
Then the **infinite part of the spectrum** $\{\lambda_{p+1}, ...\}$ **of** $P(\lambda)$ **will remain unchanged.**

Part (ii) (Assignment Part).

- **Solve the Sylvester equation**:

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (v_1, b_1) & \dots & (v_1, b_m) \\ \vdots & & \\ (v_p, b_1) & \dots & (v_p, b_m) \end{pmatrix}.$$

- Compute

$$\Phi Z_1 = \Gamma,$$

Result:

$$\Omega(P_{c1}(\lambda)) = \{\mu_1, ..., \mu_p, \lambda_{p+1}, ..., ...\}$$

.

Algorithm. **(Parametric Solution to the Partial Eigenvalue Assignment Problem in Distributed Parameter System)**

Inputs:

(a) The differential operators M, C, and K of the open-loop pencil $P(\lambda)$.

(b) The m control functions $b_1, ..., b_m$.

(c) The set of scalars $\{\mu_1, ..., \mu_p\}$, closed under complex conjugation.

(d) The self-conjugate subset $\{\lambda_1, ..., \lambda_p\}$ of the open - loop spectrum $\{\lambda_1, \lambda_2, ...\}$ and the associated eigenfunction set $\{v_1, ..., v_p\}$.

Outputs:

The feedback functions $f_1, ..., f_m$ and $f_{21}, ..., f_{2m}$ such that the spectrum of the closed-loop operator pencil is the set $\{\mu_1, ..., \mu_p;\ \lambda_{p+1}, \lambda_{p+2}, ...\}$.

Assumptions:

- The control functions $b_1, ..., b_m$ are linearly indepen- dent.
- The open-loop quadratic operator pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ with control functions $b_1, ..., b_m$ is partially controllable with respect to the eigenvalues $\lambda_1, ..., \lambda_p$.
- The sets $\{\lambda_1, ..., \lambda_p\}$, $\{\lambda_{p+1}, \lambda_{p+1}, ...\}$, and $\{\mu_1, ..., \mu_p\}$ are disjoint.
- The open-loop operator pencil $P(\lambda)$ has a discrete spectrum **without finite accumulation points,** every eigenvalue is **Semi-simple**, and the system of eigenfunctions of $P(\lambda)$ is **two-fold complete**.

  (Large Body of Literature on **Spectral Theory of Operators**).

Step 1. Form $\Lambda_1 =$ diag $(\lambda_1, ..., \lambda_p)$ and $\Lambda_{c1} =$ diag $(\mu_1, ..., \mu_p)$.

Step 2. Choose arbitrary $m \times 1$ vectors $\gamma_1, ..., \gamma_p$ in such a way that $\overline{\mu_j} = \mu_k$ implies $\overline{\gamma_j} = \gamma_k$ and form $\Gamma = (\gamma_1, ..., \gamma_p)$.

Step 3. Solve the $m \times m$ Sylvester equation for Z_1:

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (v_1, b_1) & \dots & (v_1, b_m) \\ \vdots & \ddots & \vdots \\ (v_p, b_1) & \dots & (v_p, b_m) \end{pmatrix} \Gamma.$$

If Z_1 is ill-conditioned, then return to Step 2 and select different $\lambda_1, ..., \lambda_p$.

Step 4. Solve the $m \times m$ linear system: $\Phi Z_1 = \Gamma$ for $\Phi = (\phi_{ij})$.

Step 5. If none of the $\lambda_1, ..., \lambda_p$ is zero, form for all $k = 1, ..., m$

$$f_{1k} = \sum_{j=1}^{p} \overline{\phi}_{kj} M^* v_j, \text{ and}$$

$$f_{2k} = -\sum_{j=1}^{p} (\overline{\phi}_{kj} / \bar{\lambda}_j) K^* v_j,$$

otherwise, form for all $k = 1, ..., m$,

$$f_{1k} = \sum_{j=1}^{p} \bar{\phi}_{kj} M^* v_j, \text{ and}$$

$$f_{2k} = \sum_{j=1}^{p} \bar{\phi}_{kj} (\bar{\lambda}_j M^* v_j + C^* v_j).$$

Distinguished Practical Features

- Only a small finite part of the infinite spectrum (and the associated eigenfunctions) needed to numerically implement the algorithm.
- Mathematical guarantee of **no spill-over**.
- An infinite-dimensional control problem solved using finite-dimensional control and numerically viable finite computational techniques.
- The algorithm is **parametric** in nature. This property can be exploited in designing a **numerically robust feedback control.**

Case Study With Finite Dimensional Problem

Vibration of Rotating Axel in a Power Plant

Mathematical Model: $P(\lambda) = \lambda^2 M + \lambda D + K$

- $M = \text{diag}\,(m_1, m_2, \ldots, m_n)$.
- $D =$ Symmetric tridiagonal
- $K =$ Symmetric tridiagonal

Set $\gamma_0 = \gamma_n = \kappa_0 = \kappa_n = 0$

$$D = (d_{ij}), \text{ where } d_{ij} = \begin{cases} -\gamma_i & , \quad i+1=j \\ \gamma_{i-1} + \delta_i + \gamma_i & , \quad i=j \\ -\gamma_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$K = (k_{ij}), \text{ where } k_{ij} = \begin{cases} -\kappa_i & , \quad i+1=j \\ \kappa_{i-1} + \kappa_i & , \quad i=j \\ -\kappa_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

A Benchmark Example

$$n = 111$$

- **The open-loop Eigenvalues** (222 Eigenvalues)

$$\lambda_1 = -1.3734 \times 10^{-6}$$

(The Most Unstable Eigenvalue)

$$R_e(\lambda_j) \leq -0.016267, \ j = 2, 3, \ldots, 422.$$

(Better Stability Property)

The largest contribution to the shape of the transient response is generated by the eigenvectors corresponding to λ_1.

$\lambda_1 \Longrightarrow \mu_1 = -0.016$ **(vibration will be suppressed** 10^3 **fold)**

$$x_1 \Longrightarrow \frac{1}{\sqrt{211}}(1, 1, \ldots, 1)^T = y_1.$$

The control matrix

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \\ 0 & 0 & \cdots & 1 \end{pmatrix}^T$$

Γ = parametric matrix

$= (-0.51454,\ -0.85747)^T.$

Experimental Results

- λ_1 was assigned to μ_1 accurately
- x_1 was assigned to y_1 accurately
- 2-Norm difference between the open-loop and closed-loop eigenvalue is about 1.7×10^{-6}
- $||F_1|| < 116, \; ||F_2|| < 22$
- $\dfrac{||F_1||}{||D||_2} < 0.57$ and $\dfrac{||F_2||_2}{||K||_2} < 15.10^{-11}$

(**Small Feedback Norms Desirable for Robustness**)

Conclusion

The Vibrations of the rotating turbine axel are suppressed nearly 10^3 - fold by using small feedback control forces generated by the Algorithm.

Finite Element Model Updating Problem:

Given

1. The finite element generated symmetric matrices M, K, and D:

$$M = M^T > 0, \; K = K^T \geq 0 \text{ and } D = D^T$$

2. A set of measured eigenvalues $\{\mu_1, \ldots, \mu_m\}$ and the eigenvectors $\{y_1, \ldots, y_m\}$ from a real-life structure.

Find the updated **symmetric updates** $\tilde{M}, \tilde{K}$, and $\tilde{D}$ such that

- FEM Eigenvalues $\longrightarrow$ Measured Eigenvalues
- FEM Eigenvectors $\longrightarrow$ Measured Eigenvectors
- Remaining Eigenvalues and Eigenvectors $\equiv$ No Change.

Finite Element Model Updating (FEMU)

Finite Element Model
$M = M^T \geq 0$
$K = K^T \geq 0$
$D = D^T$

$\xrightarrow{\text{ANSYS, NASTRAN}}$

$\{\lambda_1, ..., \lambda_p\}$ and $\{x_1, ..., x_p\}$

Natural Frequencies (Eigenvalues)

Mode Shapes (Eigenvectors)

Real-Life Structure
Automobile
Boeing 777

$\longrightarrow$

$\{\mu_1, ..., \mu_p\}$ and $\{y_1, ..., y_p\}$

Measured Eigenvalues

Measured Eigenvectors

FEMU: $M \longrightarrow \tilde{M} = (\tilde{M})^T = M + \Delta M$ (**Symmetric**)

$K \longrightarrow \tilde{K} = (\tilde{K})^T = K + \Delta K$ (**Symmetric**)

$D \longrightarrow \tilde{D} = (\tilde{D})^T = D + \Delta D$ (**Symmetric**)

$\{\lambda_1, ..., \lambda_p\} \longrightarrow \{\mu_1, ..., \mu_p\}$

$\{x_1, ..., x_p\} \longrightarrow \{y_1, ..., y_p\}$

$\{\lambda_{p+1}, ..., \lambda_{2n}\} \rightarrow \{\lambda_{p+1}, ..., \lambda_{2n}\}$ (**No Change**)

$\{x_{p+1}, ..., x_{2n}\} \longrightarrow \{x_{p+1}, ..., x_{2n}\}$ (**No Change**)

Difficulties

- **Finite-Element Models of very High-order.**

  Model Size Needs to be Reduced **(Model Reduction)**

- **Difficult to check no spill-over property computationally or Experimentally.**

- **Incomplete Measured Data.**

(Hard-wire Limitation)

Analytical Eigenvectors of Full-Length
Vs
Short Measured Eigenvectors.

Missing Entries Need to be Supplied.

- **Complex Data**

Real Finite Element Data
Vs
Complex Measured Data From Real-life Structures.

Challenges

- Problem should be solved without **Model Reduction** or reduction to condensed forms.

- Algorithms should be able to cope up with **Incomplete Measured and Complex Data**

- No spill-over phenomenon to be guaranteed **mathematically**.

- Algorithms should use only the available **small subset of the eigenvalues and eigenvectors** of the quadratic pencil, and the measured data.

The Current Status of the Problem

- The problem well-studied and still very much active work going on in Vibrating Industries
- Several hundred papers and a book (**Finite Element Model Updating in Structural Dynamics** by M.I. Friswell and J.E. Mottershead, 1995).
- Many Adhoc solutions by Industries (sometimes **Not Based on Sound Mathematical Reasoning**)
- Problem **Not Solved** in desirable way

Existing Techniques of Model Updating and Drawbacks

- The so-called optimization-based **Direct Methods** deal with **Linear model**:

$$P_i(\lambda) = \lambda M - K$$

rather than the **Quadratic Model**:

$$P_Q(\lambda) = \lambda^2 M + D + K.$$

- **Can not guarantee the no spill-over property.**

"The updated mass and stiffness matrices have little physical meaning and can to be related to physical changes to the finite-element model in the original model," **Friswell and Mottershead.**

Most Recent Developments

- (B.N. Datta) *Finite Element Model Updating, Eigenstructure Assignment, and Eigenvalue Embedding for Vibrating Systems*, J. Mechanical Vibration and Signal Processing (2003).

- *Ph.D Thesis* of João Carvalho, NIU 2002.

(The State-of-the-Art-Result on FEMU)

- **Symmetric Eigenvalue Embedding Approach** (Carvalho, B.N. Datta, W.W. Lin and J.N. Wang)

Available at the website:

www.math.niu.edu/∼dattab

Finite-Element Model Updating in Undamped Model

(Carvalho '2002).

- The problem **Completely Solved** in the case of Undamped Model
- The difficulties with incomplete measured data resolved in the algorithm itself.

PART I (Updating of K with No Spill-over)

Λ = The Finite Element Matrix of Eigenvalues.

X = The Finite Element Matrix of Eigenvectors.

Partition

$\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$:

$\Lambda_1 = \text{diag}\{\lambda_1, \ldots, \lambda_p\}$

$\Lambda_2 = \text{diag}\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$

$X = (X_1, X_2) : X_1 = \{x_1, \ldots, x_p\}, X_2 = \{x_{p+1}, \ldots, x_{2n}\}.$

Theorem

Let

$$\tilde{K} = K - MX_1\Phi X_1^T M.$$

Then if Φ is a **symmetric matrix**,

$$(i) \quad \tilde{K} \text{ is a symmetric matrix}$$

and

$$(ii) \quad MX_2\Lambda_2 + \tilde{K}X_2 = 0$$

$\Longrightarrow$ **No Spill-over**.

PART II (Assignment of Measured Data)

Σ = The Matrix of Measured Eigenvalues

Y_1 = Matrix of Measured Eigenvectors

Theorem Let Φ satisfy the Sylvester matrix equation:

$$(Y_1^T M X_1)\Phi(Y_1^T M X_1) = Y_1 M Y_1 \Sigma + Y_1^T K Y_1.$$

- Then Φ is **symmetric**

- $\Omega(\lambda^2 M + \tilde{K})$ = { **Measured eigenvalues**; $\lambda_{p+1} \ldots, \lambda_{2n}$}
- Eigenvectors of $(\lambda^2 M + \tilde{K})$: {**Measured eigenvectors**; $x_{p+1} \ldots, x_{2n}$}.

Notes: Y_1 = Measured Eigenvector Matrix

= **Not Completely Known**

$$= \begin{pmatrix} Y_{11} \longleftarrow \text{ Known} \\ Y_{12} \longleftarrow \text{ Unknown} \end{pmatrix}$$

- The unknown part is computed appropriately by the Algorithm.

Model Updating of an Undamped Symmetric Positive Semidefinite Model Using Incomplete Measured Data

Input: The symmetric matrices $M, K \in \mathbb{R}^{n\times n}$; the set of m analytical frequencies and mode shapes to be updated; the complete set of m measured frequencies and model shapes from the vibration test.

Output: Updated stiffness matrix $\tilde{K}$.

Assumption: $M = M^T \geq 0$ and $K = K^T \geq 0$.

Step 1: Form the matrices $\Sigma_1^2 \in \mathbb{R}^{m\times m}$ and $Y_{11} \in \mathbb{R}^{m\times m}$ from the available data. form the corresponding matrices $\Lambda_1^2 \in \mathbb{R}^{m\times m}$ and $X_1 \in \mathbb{R}^{n\times m}$.

Step 2: Compute the matrices $U_1 \in \mathbb{R}^{n\times m}$, $U_2 \in \mathbb{R}^{n\times(n-m)}$, and $Z \in \mathbb{R}^{m\times m}$ from the QR factorization:

$$MX_1 = [U_1\ U_2]\begin{bmatrix} Z \\ 0 \end{bmatrix}$$

Step 3: Partition $M = [M_1\ M_2], K = [K_1\ K_2]$ where $M_1, K_1 \in \mathbb{R}^{n\times m}$.

Step 4: Solve the following matrix equation to obtain $Y_{12} \in \mathbb{R}^{(n-m)\times m}$:

$$U_2^T M_2 Y_{12} \Sigma + U_2^T K_2 Y_{12} = -U_2^T \left[K_1 Y_{11} + M_1 Y_{11} \Sigma \right]$$

and form the matrix

$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}.$$

Theorem on Symmetry Preserving Partial Eigenvalue Assignment

Let (λ_1, y_1) *be an unwanted real isolated eigenpair of* $P(\lambda) = \lambda^2 M + \lambda D + K$ *with* $y_1^T K y_1 = 1$. *Let* λ_1 *be reassigned to* μ_1. *Define* $\theta_1 = y_1^T M y_1$ *and assume that* $1 - \lambda_1 \mu_1 \theta_1 \neq 0$ *and* $1 - \lambda_1^2 \theta_1 \neq 0$.

- (λ_1, Y_1) - An Unwanted Isolated Real Eigenpair
- $\theta_1, = y_1^T M Y_1$
- $\in = \dfrac{\lambda_1 - \mu!}{1 - \lambda_1 \mu_1 \theta}$
- Updated model $P_U(2) = x^2 M_U + \lambda D_U + K_U$

is such that

$$M_U = M - \epsilon_1 \lambda_1 M y_1 y_1^T M$$

$$D_U = D + \epsilon_1 (M y_1 y_1^T K + K y_1 y_1^T M)$$

$$K_U = K - \frac{\epsilon_1}{\lambda_1} K y_1 y_1^T K$$

i. The eigenvalues of $P_U(\lambda)$ the same as those of $P(\lambda)$ except that λ_1 replaced by μ_1.

ii. y_1 also an eigenvector of $P_U(\lambda)$ corresponding to the embedded eigenvalue μ_1.

iii. If (λ_2, y_2) an eigenpair of $P(\lambda)$, where $\lambda_2 \neq \lambda_1$, then (λ_2, y_2) also an eigenpair of $P_U(\lambda)$.

Conclusions

- Some very interesting (but **very difficult**) **Structured Inverse Eigenvalue Problems** arising in practical Industrial Applications.

- **Real-life applicable** and **mathematically sound** solutions.

- Many existing industrial techniques are **ad-hoc** in nature. *Not much consideration for mathematical difficulties and challenges.*

- Very often *lacks strong mathematics foundations.*

- Industries in **Japan** and **Germany** take **more mathematical approach to industrial problems.**

- Need people with **industrial aptitude and interdisciplinary training** blending *Linear Algebra, Numerical Linear Algebra, and Scientific Computing* with areas of engineering such as *Mechanical and Electrical Engineering.* **Such expertise are rare.**

- Curricular in both **Engineering, Mathematics and Computer Science** need to be re-looked into for opportunities for **interdisciplinary courses.**

- Many **engineering text books** need to be rewritten incorporating recent developments in **matrix computations, scientific computing and mathematical software**.