Some things I learned from Douglas Munn and some things I still need to learn

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Characteristics of Munn's work

1. the concepts involved are clearly and intuitively expressed
2. the results obtained are clearly stated and easy to understand
3. the proofs appear simple - so simple that it is easy to forget that this simplicity and clarity was the result of a great deal of work on Douglas' part
    - the proofs appear simple because exactly the right concepts had been introduced to make the proofs appear simple;
    - the choice of the concepts was the result of Douglas' incredible insight into the situation in question;
    - as well as insight, the proofs in their final form are the end product of a process of a lot of rewriting and polishing until the final form is just about perfect.
4. the results in his papers lead to new applications, insights, and questions.

ω-regular semigroups

An inverse semigroup S is said to be *ω-regular* if the semilattice E of idempotents is isomorphic to the chain of natural numbers under the reverse of the usual partial order on the natural numbers.
Thus we can enumerate $E(S)$ as e_n, $n \geq 0$ where

$$e_0 > e_1 > e_2 > \cdots > e_n > \cdots .$$

We call such a chain an *ω-chain*.

Theorem

(N.R. Reilly in his Ph.D.thesis under Munn) Let G be a group and θ an endomorphism of G. Then $B(G,\theta) = \mathbb{N}^0 \times G \times \mathbb{N}^0$ is a bisimple ω-regular semigroup under the multiplication

$$(m,g,n)(r,h,s) = (r \vee n - n + m, g\theta^{r\vee n-n}h\theta^{n\vee r-r}, n \vee r - r + s)$$

where $\vee = \max$. Conversely, each bisimple ω-regular semigroup is isomorphic to a semigroup of the form $B(G,\theta)$.

ω-regular semigroups

Let $G_0, \cdots, G_{d-1}$ be a chain of d-groups, linked together by homomorphisms $G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_{d-1}$. Then we can use these linking homomorphisms to turn $\mathcal{G} = G_0 \cup \cdots G_{d-1}$ into an inverse semigroup which is a union of groups.

Theorem

(Munn and independently Kochin) Let $\mathcal{G} = G_0 \cup \cdots \cup G_{d-1}$ *be a finite chain of groups and* θ *be an endomorphism of* $\mathcal{G}$ *such that* $\mathcal{G}\theta \subseteq G_0$*. Then* $B(\mathcal{G}, \theta) = \mathbb{N}^0 \times \mathcal{G} \times \mathbb{N}^0$ *is a simple* ω*-regular semigroup with* d $\mathcal{D}$*-classes under the multiplication*

$$(m, g, n)(r, h, s) = (r \vee n - n + m, g\theta^{r \vee n - n} h\theta^{n \vee r - r}, n \vee r - r + s).$$

Each simple ω*-regular semigroup is isomorphic to* $B(\mathcal{G}, \theta)$ *for some finite chain of* d *groups and some endomorphism* θ *of* $\mathcal{G}$ *such that* $\mathcal{G}\theta \subseteq G_0$ *and some positive integer* d*.*

Tiling semigroups - Kellendonk, 1995-1997

An *n-dimensional tiling is a covering of* $\mathbb{R}^n$ by closed bounded and connected subsets of $\mathbb{R}^n$ with connected interior and which are the closures of their interiors, called *tiles*, which are disjoint except possibly at the boundary of the tiles.

A *pattern* is a finite connected union of tiles with connected interior.

A *pattern class* is the equivalence class of a pattern under translation and a *doubly pointed pattern class* is a pattern class with the choice of an ordered pair of tiles, usually called *in-tile* and *out-tile* respectively.

An n-dimensional tiling is said to be of *finite type* if there are only a finite number of tiling patterns whose underlying pattern has two tiles. All the tilings we shall consider are of finite type. Note that this means that there are only a finite number of tiling patterns with just one tile.

Tiling semigroups - Kellendonk, 1995-1997

The doubly pointed pattern classes of a tiling $\mathcal{T}$, together with a symbol 0, can be turned into an inverse semigroup $S(\mathcal{T})$.

Intuitively, given two doubly pointed pattern classes A and B, if there exist $(a, P, c) \in A$ and $(c, Q, b) \in B$ such that $P \cup Q$ is a pattern of the tiling, then $A * B$ is the class of $(a, P \cup Q, b)$; otherwise $A * B = 0$.

One-dimensional tiling semigroups

From an algebraic point of view, a one-dimensional tiling $\mathcal{T}$ can be identified with a doubly infinite word w over the alphabet Σ of one-tile pattern classes.

The pattern classes then correspond to the members of Σ^+ which are subwords of w. We call this set of words the *language* $L(\mathcal{T})$ of $\mathcal{T}$.

Lawson shows that the non-zero members of the tiling semigroup $S(\mathcal{T})$ can then be identified with the symbols

$$\begin{aligned} &a_1 a_2 \cdots \grave{a}_i \cdots \acute{a}_j \cdots a_n, \\ &a_1 a_2 \cdots \acute{a}_i \cdots \grave{a}_j \cdots a_n, \\ &\text{and } a_1 a_2 \cdots \check{a}_i \cdots a_n \end{aligned}$$

where $a_1 a_2 \cdots a_i \cdots a_n \in L(\mathcal{T})$;

the grave accent indicates the in-tile; the acute accent denotes the out-tile; the $\check{}$ indicates that the in-tile and out-tile are the same (thus we have an idempotent).

One-dimensional tiling semigroups

Multiplication of non-zero elements is carried out as if we were dealing with linear Munn trees.

Clearly, instead of using accents, we could indicate the in-tile and the out-tile by indicating their indices in the underlying word $a_1 a_2 \cdots a_i \cdots a_j \cdots a_n \in L(\mathcal{T})$.

In this way we get a representation of the non-zero elements of $S(\mathcal{T})$ as the set of all triples $(i, u, j) \in \mathbb{N}^0 \times L(\mathcal{T}) \times \mathbb{N}^0$, with $1 \leq i, j \leq |u|$.

One-dimensional tiling semigroups have been studied by Kellendonk and Lawson (2000), Lawson (2004), and also by Filipa Soares and McA (2009).

n-dimensional hypercubic tiling semigroups

n-dimensional hypercubic tilings

An n-hypercubic tiling is a tiling of $\mathbb{R}^n$ by hypercubes of side 1 (that is, translations of $[0,1]^n$), possibly colored, and aligned in such a way that the center of each tile belongs to $\mathbb{Z}^n$.

This means that adjacent tiles share a common face.

One-dimensional tilings of finite type can be considered as one-dimensional hypercubic tilings: instead of regarding the tiling as a collection of intervals of (possibly) different lengths, consider the tiles as being colored intervals of length 1.

Σ-colored subsets of $\mathbb{Z}^n$

Let Σ be a (finite) set of colors.

Then a Σ-*colored subset* of $\mathbb{Z}^n$ is a pair (A, g) where A is a subset of $\mathbb{Z}^n$ and g is a map $A \to \Sigma$; $g(a)$ gives the color of each $a \in A$.

We say that (A, g) is finite, or non-empty, if A is finite, or non-empty.

It is *connected if* A is the set of vertices of a connected subgraph of the Cayley graph of $\mathbb{Z}^n$ with respect to the usual set of generators.

There is an obvious partial order on the on the set of Σ-colored subsets of $\mathbb{Z}^n$: $(A, g) \leq (B, h)$ if and only if $A \subseteq B$ and $h(a) = g(a)$ for each $a \in A$.

Under this partial order, (A, g) and (B, h) have a common upper bound if and only if g and h agree on $A \cap B$. In this case, the least upper bound $(A, g) \cup (B, h) = (A \cup B, k)$ where $k(c) = g(c)$ if $c \in A$ while $k(c) = h(c)$ if $c \in B$.

Σ-colored subsets of $\mathbb{Z}^n$ (continued)

The group $\mathbb{Z}^n$ acts on the set of Σ-colored subsets of $\mathbb{Z}^n$ by translation:

$$(A, g) + u = (A + u, h) \text{ where } h(a + u) = g(a) \text{ for each } a \in A.$$

The color of the image under the action is the same as the color of the original member of A.

The action clearly preserves finiteness and connectedness.

A *hypercubic tiling* $\mathcal{T}$ of $\mathbb{Z}^n$ is a Σ-coloring $(\mathbb{Z}^n, f)$.

A tiling pattern of $\mathcal{T}$ is then a finite connected Σ-colored subset $(A, g) \leq (\mathbb{Z}^n, f)$.

The coloring on A is the restriction f_A of f to A and so is determined by A. We shall often omit explicit mention of the coloring map of A.

The language of a hypercubic tiling

The lexicographic ordering on $\mathbb{Z}^n$ is defined by

$(a_1, \cdots , a_n) \leq (b_1, \cdots , b_n) \Leftrightarrow \exists r \geq 0$ s.t. $a_r < b_r$ and $a_i = b_i$, $i > r$.

This is a linear ordering which is compatible with the addition in $\mathbb{Z}^n$. Because it is a linear ordering, every finite subset A of $\mathbb{Z}^n$ has a smallest member which we will normally denote by a_0.

The language of a hypercubic tiling

The subset $A = \{1001, 1002, 1003, 1004, 1005, 1006\}$ of $\mathbb{Z}^1$ has $a_0 = 1001$. If a coloring is given by the table

$$\left(\begin{array}{cccccc} 1001 & 1002 & 1003 & 1004 & 1005 & 1006 \\ a & b & c & d & e & f \end{array} \right)$$

then $A - a_0$ is the colored subset given by the table

$$\left(\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e & f \end{array} \right)$$

which represents the word $a_0 a_1 a_2 a_3 a_4 a_5 = abcdef$.

So the language $L(\mathcal{T})$ of a one-dimensional tiling is precisely the set $\{A - a_0 : A \text{ is a tiling pattern in } \mathcal{T}\}$.

The language of a hypercubic tiling

Definition

Let $\mathcal{T}$ be an *n*-dimensional hypercubic tiling. Then the language $L(\mathcal{T})$ of $\mathcal{T}$ is the set $\{A - a_o : A \text{ is a pattern in } \mathcal{T}\}$.

Thus the language of $\mathcal{T}$ consists of all translations of tiling patterns whose smallest member is the zero 0 of $\mathbb{Z}^n$.

The language is factorial in the sense that $A \in L$ implies that $B - b_0 \in L$ for each finite connected subset B of A.

In the one-dimensional case this is equivalent to saying that "$w \in L$ implies each subword of w belongs to L".

Hypercubic tiling semigroups

Lemma

Let L be a factorial n-dimensional hypercubic language over Σ. Then the set $S(L) = \{(a, A, b) : A \in L \text{ and } a, b \in A\} \cup \{0\}$ is a strongly E^*-unitary inverse semigroup under the multiplication

$$(a, A, b)(c, C, d) = (c \vee b - b + a, P, b \vee c - c + d)$$

when

$$\begin{aligned} P &= [A + (c \vee b - b)] \cup [C + (b \vee c - c)] \\ &= [(A - b) \cup (C - c)] + c \vee b \in L, \end{aligned}$$

and all other products equal to 0.

Theorem

Let $\mathcal{T}$ be an n-dimensional hypercubic tiling semigroup with language L. Then $S(\mathcal{T}) \approx S(L)$.

Hypercubic tiling semigroups

Theorem

Let $\mathcal{T}$ *be an n-dimensional hypercubic tiling semigroup with language L. Let* $\mathcal{X} = \mathbb{Z}^n \times L$ *and* $\mathcal{Y} = \{(p, P) : p \in P\}$*. Then defining*

$$(p, P) \leq (q, Q) \Leftrightarrow Q - q \subseteq P - p$$

for all $(p, P), (q, Q) \in \mathcal{X}$*, and the action of* $\mathbb{Z}^n$ *on* $\mathcal{X}$ *by* $u.(p, P) = (p - u, P)$*, for all* $u \in \mathbb{Z}^n$ *and* $(p, P) \in \mathcal{X}$*, we have that* $(\mathbb{Z}^n, \mathcal{X}, \mathcal{Y})$ *is a* P^**-triple and* $S(\mathcal{T}) \approx P^*(\mathbb{Z}^n, \mathcal{X}^0, \mathcal{Y}^0)$*.*

Hypercubic tiling semigroups

Theorem

Let $\mathcal{T}$ be a one-dimensional tiling with language L. Let $\mathcal{X} = \mathbb{Z} \times L$ and $\mathcal{Y} = \{(i, w) : 1 \leq i \leq |w|\}$. Then defining

$$(i, w) \leq (j, u) \Leftrightarrow w = xuy \text{ where } |x| = i - j$$

and the action of $\mathbb{Z}$ on $\mathcal{X}$ by $u.(i, w) = (i - u, P)$, for all $u \in \mathbb{Z}$ and $(i, w) \in \mathcal{X}$, we have that $(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ is a P^*-triple and $S(\mathcal{T}) \approx P^*(\mathbb{Z}, \mathcal{X}^0, \mathcal{Y}^0)$.