

Transformation Semigroups: Congruences, Idempotents, and Groups

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Conference for **young** algebraists.

1961

The Algebraic Theory of Semigroups, Vol. 1

A. H. Clifford and G. B. Preston

Math. Surveys of the American Mathematical Society

1967

The Algebraic Theory of Semigroups, Vol. 2

1962-1964

Papers of H-J Höehnke

“the bulk of Höehnke’s work relies on ring theory and is based on the observation that, in most aspects, congruences on semigroup play the role of ideals of rings.

Moreover, in the introduction to the paper “Zur Strukturtheorie der Halbgruppen”, Math. Nachrichten, **26** (1963), 1-13 he (Höehnke) outlines an ambitious program as follows:

The natural starting point for the structure theory of semigroups is the theory of transformation semigroups, in parallel to Jacobson’s structure theory for rings which uses linear transformations of vector spaces.

The program proved to be overly optimistic

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Höehnke did manage, however, to carry over Jacobson’s theory of primitive rings, first to semigroups with zero and then to general semigroups, and he elaborated some of the more particular aspects of the theory.”

Jacobson's Density Theorem

Let R be a ring which admits a faithful simple R -module M ; that is R is a primitive ring. Then $D = \text{End}_R(M)$ is a division ring.

Let A be any D -linear operator on M and let X be any finite D -linearly independent subset of M . Then there exists an element $r \in R$ such that $xA = x.r$ for all $x \in X$.

R is isomorphic to a dense ring of linear transformations of a vector space over D .

R has a high degree of transitivity when represented as a ring of linear operators over D .

\mathcal{T}_X : all functions $\alpha : X \rightarrow X$ on a non-empty set X , written on the right;

\mathcal{T}_X is a semigroup under composition of functions; it is a monoid with identity the function 1 defined by $x1 = x$ for all $x \in X$;

the group of units is the full symmetric group \mathcal{S}_X on X ;

$\varepsilon \in \mathcal{T}_X$ is idempotent if and only if it fixes each element of $X\varepsilon$;
 $x\varepsilon = x$ for each $x \in X\varepsilon$;

the constant maps c_a , $a \in X$ are idempotents;

if $|X| = n$ is finite then \mathcal{T}_X has n^n elements; $\sum_{r=1}^n \binom{n}{r} r^{n-r}$ are idempotents;

\mathcal{T}_X is regular.

For any $\alpha \in \mathcal{T}_X$, $X\alpha$ and the set of equivalence classes $X/\alpha \circ \alpha^{-1}$ have the same cardinality. We call this cardinal the *rank* of α and, for each cardinal $n \leq |X|$, we write $I_n = \{\alpha \in \mathcal{T}_X : \text{rank}(\alpha) < n\}$. For convenience, if $n = |X|$ we write $I_{n+1} = \mathcal{T}_X$.

Proposition

Let $\alpha, \beta \in \mathcal{T}_X$. Then

$\alpha \mathcal{L} \beta$ if and only if $X\alpha = X\beta$;

$\alpha \mathcal{R} \beta$ if and only if $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

$\alpha \mathcal{D} \beta$ if and only if $\text{rank}(\alpha) = \text{rank}(\beta)$ if and only if $\alpha \mathcal{J} \beta$;

$\alpha \leq_{\mathcal{J}} \beta$ if and only if $\text{rank} \alpha \leq \text{rank} \beta$.

Proposition

The ideals of \mathcal{T}_X form a chain under inclusion. Indeed

the proper ideals of \mathcal{T}_X are exactly the sets I_n , $n \leq |X|$;

the principal factors $J_n = I_{n+1}/I_n$ are all 0-bisimple regular semigroups;

if n is finite, J_n is completely 0-simple.

Congruences on full transformation semigroups

- A.I. Malcev, Symmetric Groupoids (Russian), Mat. Sbornik, **31** (1952), 136-151. (English translation: American Math. Soc. Translations, **113** (1979), 235-250 by Heidi Semla and Bob Sullivan)
- A. H. Clifford and G.B. Preston, Algebraic Theory of Semigroups, Vol.2 Math, Surveys of the AMS, 7, 1967.
- R.P. Sullivan, Transformation Semigroups: past, Present, and Future, 191-243, Proceedings of The International Conference on Semigroups, Braga, Portugal 1999, World Scientific, 2000.

Congruences on full transformation semigroups

We assume X is either countable or finite with more than 2 members and that ρ is a non-trivial and non-universal congruence on \mathcal{T}_X .

- 1 All constant maps belong to the same ρ -class, which is an ideal.

$$I_\rho = \{\alpha \in \mathcal{T}_X : (\alpha, c_a) \in \rho \text{ for some } a \in X\}$$

is an ideal which contains all the constant maps.

- 2 $I_\rho = I_n$ for some $n \geq 2$ or else, if $|X|$ is countable, $n = \omega = |X|$.
That is, either $I_\rho = I_n$ for some $n \geq 2$ or I_ρ consists of all $\alpha \in \mathcal{T}_X$ of finite rank (in case $|X| = \omega$).
- 3 ρ can be factored through the Rees congruence on \mathcal{T}_X corresponding to the ideal I_ρ so it suffices to describe the 0-restricted congruences on these semigroups.

Case 1: finite zero ideal

$I_\rho = I_n$ with n finite. \mathcal{T}_X/I_ρ has a unique 0-minimal ideal which is a completely 0-simple semigroup isomorphic to the principal factor J_n .

- 1 Every non-trivial congruence on J_n is contained in \mathcal{H} , that is it is idempotent separating..

This follows from the “denseness” of \mathcal{T}_X in the sense of Jacobson.

- 2 Any idempotent separating congruence η on J_n can be extended to a congruence on \mathcal{T}_X/I_n .

This depends only on the fact that the semigroup is regular and has a unique 0-minimal ideal.

- 3 The extension can be carried out in exactly one way. It is defined as

$$\alpha \equiv \beta \text{ if and only if } \alpha = \beta \text{ or } \alpha, \beta \in J_n \text{ and } (\alpha, \beta) \in \eta.$$

Again, this is due to the density of \mathcal{T}_X .

- 4 Finally, the idempotent separating congruences on J_n are determined by the normal subgroups of any of its maximal (non-zero) subgroups. These subgroups are isomorphic to the symmetric group \mathcal{S}_n .

Case 2: infinite zero ideal

Let $\alpha, \beta \in \mathcal{T}_X$ with X countably infinite.

$$D(\alpha, \beta) = \{x \in X : x\alpha \neq x\beta\}.$$

If $\alpha = \beta$ then $d(\alpha, \beta) = 0$;

Otherwise $d(\alpha, \beta) = \max(|D(\alpha, \beta)\alpha|, |D(\alpha, \beta)\beta|)$.

The relation δ on J_ω defined by, for α, β of infinite rank, by

$(\alpha, \beta) \in \delta$ if and only if $d(\alpha, \beta)$ is finite

and $(0, 0) \in \delta$, is a non-identity, and non-trivial congruence on J_ω .

It is the unique such congruence. Again, this depends on the denseness of \mathcal{T}_X .

Generators of full transformation semigroups

Theorem

(Howie, 1966) Let X be a finite set. with n members. Then every $\alpha \in \mathcal{T}_X$ which is not a permutation is a product of idempotents of rank $n - 1$.

Corollary

The semigroup of all singular (=non-permutation) transformations is regular and idempotent generated.

Proposition

(Ruškuc) Let G be a group of permutations on X and let e be an idempotent of rank $n - 1$. Then $S = \langle G, e \rangle$ contains all singular transformations if and only if G acts weakly doubly transitively on X ; that is, G acts transitively on the 2-element subsets of X .

Theorem

Let X be a finite set of order n and let e be an idempotent transformation of rank $n - 1$. Then $S = \langle G, e \rangle$ is regular.

For convenience, suppose that $e = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the idempotent that maps 1 to 2 and leaves the other elements of X fixed. The proof divides into two cases:

- 1 2 does not belong to the orbit of 1 under the action of G .
In this case the semigroup $S = \langle G, e \rangle$ is orthodox.
- 2 2 does belong to the orbit of 1.

Theorem

$S = \langle G, e \rangle$ is inverse if and only if 2 does not belong to the orbit of 1 under the action of G and the stabilizer $S_1 = \{g \in G : 1g = 1\}$ of 1 under G is contained in $S_2 = \{g \in G : 2g = 2\}$.

The number of \mathcal{D} -classes is given by the Cauchy-Frobenius formula

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{1}{|G|} \sum_{g \in G} |\{x \in X : x.g = x\}|.$$

In special cases this can be used to find interesting structural information about S .

Example

$S = \langle G, e \rangle$ where e is an idempotent of rank less than $n - 1$.

G is the cyclic group generated by $g = (1, 3, 5)(2, 4)$ and e is the idempotent with action $(3, 3, 3, 4, 5)$. The semigroup $S = \langle G, e \rangle$ has 159 elements 12 of which are not regular.

Example

$S = \langle G, a \rangle$ where a is a non-idempotent of rank $n - 1$.

G is the cyclic group generated by $g = (1, 4, 5)(2, 6, 3)$ and a is the element (non idempotent) with action $(4, 4, 3, 2, 6, 5)$. The semigroup $S = \langle G, a \rangle$ has 3003 elements of which 288 are not regular.

Jorge André, (thesis Univ. de Lisboa, 2002 and subsequent papers) considered semigroups $S = \langle G, U \rangle$ where G is a group of permutations on a finite set with n members and U is a set of transformations of rank $n - 1$.

He is able to give some sufficient conditions for S to be regular and to generalize some of the results which hold for semigroups of the form $S = \langle G, e \rangle$ where e is an idempotent of rank $n - 1$.

The results in the general situation are much more difficult and complicated than those that hold when e is an idempotent.

Lemma

Let G be a group of permutations on a finite set X and let $a \in \mathcal{T}_X$. Then a is a regular element of $S = \langle G, a \rangle$ if and only if

$$K_G(a) = \{g \in G : \text{rank}(aga) = \text{rank}(a)\}$$

is non-empty.

Suppose that a has rank $n - 1$. Then it is easy to see that $K_G(a)$ is non-empty when G acts transitively on X .

Proposition

Let G be a group of permutation on a finite set X with n elements and let a have rank $n - 1$. Then $S = \langle G, a \rangle$ contains all singular transformations on X if and only if G acts weakly doubly transitively on X .

Corollary

Let S be a submonoid of T_X . Then S contains all singular transformations on X if

- 1 *the group of units of S acts weakly doubly transitively on X ;*
- 2 *S contains an element of rank $|X| - 1$.*

Universal transversal property

(Araújo, Mitchell, and Schneider) A permutation group G on a finite set X has the *universal transversal property* (UTP) if, for each finite subset I of X and each partition P of X with $|I|$ classes, there exists $g \in G$ such that Ig is a transversal of P .

Lemma

Let X be a finite set and G a group of permutations on X . Then G has UTP if and only if $K_G(a)$ is non-empty for each $a \in X$.

Theorem

(Araújo, Mitchell, and Schneider) Let X be a finite set and G a group of permutations on X . Then the following are equivalent:

- 1 $S = \langle G, a \rangle$ is regular for each $a \in \mathcal{T}_X$;
- 2 $S = \langle a^g = g^{-1}ag : g \in G \rangle$ is regular for each $a \in \mathcal{T}_X$; note that $\langle a^g = g^{-1}ag : g \in G \rangle$ is the subsemigroup generated by all the G -conjugates of a ;
- 3 G has UTP.

Corollary

Let X be a finite set and G a group of permutations on X . Then every submonoid of \mathcal{T}_X , with group of units G , is regular if and only if G has UTP.

Theorem

(Araújo, Mitchell, and Schneider) Let X be a finite set and G a group of permutations on X . Then G has UTP if and only if one of the following holds:

- ① $|X| = 5$ and $G \approx C_5, D_5,$ or $AGL(1, 5)$;
- ② $|X| = 6$ and $G \approx PSL(2, 5}$ or $PGL(2, 5)$;
- ③ $|X| = 7$ and $G \approx AGL(1, 7)$;
- ④ $|X| = 8$ and $G \approx PGL(2, 7)$;
- ⑤ $|X| = 9$ and $G \approx PSL(2, 8)$ or $PT(2, 8)$;
- ⑥ $G \approx \mathcal{A}_{|X|}$ or $G \approx \mathcal{S}_{|X|}$.

Theorem

(Araújo, Mitchell, and Schneider) Let X be a finite set and G a group of permutations on X . Then the following are equivalent:

- ① The semigroup $\langle G, a \rangle \setminus G$ is idempotent generated for all non-singular transformations $a \in \mathcal{T}_X$;
- ② the semigroup $\langle a^g = g^{-1}ag : g \in G \rangle$ is idempotent generated for all non-singular $a \in \mathcal{T}_X$;
- ③ one of the following holds for G
 - ① $|X| = 5$ and $G \approx \text{AGL}(1, 5)$;
 - ② $|X| = 6$ and $G \approx \text{PSL}(2, 5)$ or $\text{PGL}(2, 5)$;
 - ③ $G = \mathcal{A}_{|X|}$ or $G = \mathcal{S}_{|X|}$.

Corollary

Let X be a finite set and G a group of permutations on X . Then the ideal of non-units of each submonoid of $\mathcal{T}_{|X|}$, which has group of units G , is idempotent generated (and regular) if and only if G is one of the groups described in 3 above.