On a Class of Lattice Ordered Inverse Semigroups

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Abstract

It is well known that the free group on a non-empty set can be totally ordered and, further, that each compatible lattice ordering on a free group is a total ordering. On the other hand, Saitō has shown that no non-trivial free inverse semigroup can be totally ordered. In this note we show however that every free inverse monoid admits compatible lattice orderings which are closely related to the total orderings on free groups.

These orderings are natural in the sense that the imposed partial ordering on the idempotents coincides with the natural partial ordering. For this to happen in a lattice ordered inverse semigroup, the idempotents must form a distributive lattice. The method of construction of the lattice orderings on free inverse monoids can be applied to show that naturally lattice ordered inverse semigroups with a given distributive lattice $E$ of idempotents can have arbitrary Green’s relation structure. Analogous results hold for naturally $\Lambda$-semilatticed inverse semigroups. In this case, there is no restriction on the semi-lattice $E$ of idempotents.

We also show that every compatible lattice ordering on the free monogenic inverse monoid is of the type considered here. This permits us to prove that there are precisely eight distinct compatible lattice orderings on this semigroup. They belong to two families, each of which contains four members, of conjugate lattice orderings.
1 Introduction

An inverse semigroup $S$ is said to be a partially ordered semigroup, or to be partially ordered, if it admits a compatible partial ordering $\leq$; that is, $\leq$ is a partial order on the set $S$ and for each $a, b \in S$,

$$a \leq b \text{ implies } xay \leq xby \text{ for all } x, y \in S^1.$$

Every inverse semigroup admits at least one such partial ordering; in particular, it admits the natural partial ordering which is defined by

$$a \leq b \text{ if and only if } a = eb \text{ for some idempotent } e.$$

This partial ordering plays a crucial role in the theory of inverse semigroups. Because we will be dealing with an imposed partial order in addition to the natural partial order, we shall denote the natural partial order by $\preceq$. The imposed partial order will be denoted by $\leq$.

A partially ordered inverse semigroup $S$ is said to be a $\lor$-semilatticed semigroup if the least upper bound $a \lor b$ exists for each pair of elements $a, b \in S$ and if multiplication distributes over the join operation $\lor$; that is

$$c(a \lor b) = ca \lor cb \text{ and } (a \lor b)d = ad \lor bd$$

for each $c, d \in S$. In a dual way, we may consider $\land$-semilatticed semigroups. An inverse semigroup $S$ is a lattice ordered semigroup if it is simultaneously a $\lor$-semilatticed semigroup and a $\land$-semilatticed semigroup under the partial ordering $\preceq$.

Lattice ordered groups are defined in a similar fashion. The reader is referred to the books by Birkhoff [1], Darnel[2], or Fuchs[3] for an introduction to lattice ordered groups. However, for the sake of completeness, we have included a number of their most basic properties in the next two lemmas and their corollaries; we shall need these results later in the paper. Results on inverse semigroups can be found in the books of [5], [6], and [12].

**Lemma 1.1** Let $G$ be a partially ordered group and let $a, b \in G$. Then the following are equivalent

(i) $a \lor b$ exists;

(ii) $a \land b$ exists;
(iii) \( a^{-1} \lor b^{-1} \) exists;

(iv) \( a^{-1} \land b^{-1} \) exists.

Further, when \( a \lor b \) exists, so do \( ca \lor cb \) and \( ad \lor bd \) for each \( c, d \in G \) and then

\[
ca \lor cb = c(a \lor b) \quad \text{and} \quad ad \lor bd = (a \lor b)d
\]

for each \( c, d \in G \). In addition, \( a \land b \) can be obtained from \( a \lor b \) using either of the following formulas:

\[
a \land b = (a^{-1} \lor b^{-1})^{-1}
\]

\[
a \land b = a(a \lor b)^{-1}b.
\]

**Corollary 1.2** Let \( G \) be a partially ordered group. Then \( G \) is a lattice ordered group if and only if \( a \lor b \) [or \( a \land b \)] exists for each \( a, b \in G \).

The results in the lemma and corollary follow primarily because multiplication is an order isomorphism in any lattice ordered group while inversion is an order anti-isomorphism; that is \( a^{-1} \leq b^{-1} \) if and only if \( a \geq b \). Neither of these assertions need be valid in a partially ordered inverse semigroup and so the results of the lemma and corollary need not hold for inverse semigroups. Indeed it is possible for a partially ordered inverse semigroup to be a lattice without being either a \( \lor \)- or a \( \land \)-semilatticed semigroup.

Another consequence of the relationship between multiplication and order in a lattice ordered group, which may not be valid in inverse semigroups, is the following.

**Corollary 1.3** Every lattice ordered group is a distributive lattice.

A partially ordered inverse semigroup \( S \) (or a group) is said to be totally ordered or a totally ordered inverse semigroup if the imposed partial order is a total order; that is, if \( a, b \) are distinct elements of \( S \) then either \( a < b \) or \( b < a \). Because of the translation properties of multiplication in a lattice ordered group we have the following test for a lattice ordered group to be totally ordered.

**Lemma 1.4** Let \( G \) be a lattice ordered group. Then \( G \) is not totally ordered if and only if there are elements \( a, b \in G \), different from the identity 1 of \( G \), with \( a \land b = 1 \); equivalently, there are elements \( a, b \) distinct from 1 with \( a \lor b = 1 \).
Elements different from 1 with the property that \( a \wedge b = 1 \) are called orthogonal elements. They play an extremely important role in the theory of lattice ordered groups.

In view of the natural partial order there is an intimate relationship between the theory of inverse semigroups and the theory of groups. In particular, there is a least congruence \( \sigma \) on any inverse semigroup \( S \) for which the quotient semigroup \( S/\sigma \) is a group; \cite{10}. This congruence is given by

\[(a, b) \in \sigma \text{ if and only if } ea = eb \text{ for some idempotent } e \in S.\]

Thus, in terms of the natural partial order \( \preceq \) we have \((a, b) \in \sigma\) if and only if there exists \( c \in S \) such that \( c \preceq a, c \preceq b \) so that the \( \sigma \)-classes are the connected components of the graph of the partial order \( \preceq \). The congruence \( \sigma \) extends, in a natural way, to the situation where \( S \) is a partially ordered semigroup.

The following result may be regarded as folklore; it goes back, at least, to Saitô \cite{13}. For the sake of completeness, we give a proof of the portion concerning the case when \( S \) is a semilatticed, or latticed, inverse semigroup.

**Proposition 1.5** Let \( S \) be a partially ordered inverse semigroup and let \( G = S/\sigma \) with \( \eta \) denoting the canonical homomorphism of \( S \) onto \( G \). Then \( G \) is a partially ordered group under the partial order defined by

\[A \preceq B \text{ if and only if } a \preceq b \text{ for some } a \in A, b \in B;\]

equivalently

\[a\eta \preceq b\eta \text{ if and only if } ea \preceq eb \text{ for some } e^2 = e \in S.\]

The homomorphism \( \eta \) is order preserving; that is \( a \preceq b \) implies \( a\eta \preceq b\eta \).

If \( S \) is a semilattice ordered inverse semigroup then \( G \) is a lattice ordered group under the partial order \( \preceq \) and \( \eta \) is a semilattice homomorphism as well as an algebraic homomorphism.

**Proof.** Suppose that \( S \) is a \( \vee \)-semilattice ordered semigroup under \( \preceq \) and let \( A, B \in G \) with \( A = a\eta \) and \( B = b\eta \). Then \( A \preceq (a \vee b)\eta \) and \( B \preceq (a \vee b)\eta \). Conversely, suppose that \( A \preceq X, B \preceq X \) for some \( X = x\eta \in G \). Then there exist idempotents \( e, f \in S \) such that \( ea \preceq ex \) and \( fb \preceq fx \). Since idempotents in an inverse semigroup commute, we may deduce that
\( efa \leq ef x \) and \( ef b \leq ef x \). Hence \( ef (a \lor b) = efa \lor efb \leq ef x \) and so, from the definition of \( \leq, (a \lor b)\eta \leq x\eta = X \). Thus \( A \) and \( B \) have greatest lower bound \( A \lor B = (a \lor b)\eta \).

Since \( A, B \) are arbitrary members of \( G \), it follows, from Lemma 1.1, that \( G \) is a lattice ordered group and, clearly from the form of \( A \lor B \), \( \eta \) preserves the join operation as well as multiplication.

In a number of interesting papers T. Saitô studied the structure of totally ordered inverse semigroups; [13]. In particular, he studied those which were E-unitary in the sense that \( ea = e = e^2 \) implies \( e^2 = e \). Saitô showed that in this case the semigroup can be coordinatized by the pairs \((aa^{-1}, a\eta)\). More precisely

**Lemma 1.6** [13]Let \( S \) be an inverse semigroup. Then \( S \) is E-unitary if and only if \( \sigma \cap R = \Delta \), the identity relation on \( S \). If \( S \) is E-unitary then

\[
(a, b) \in \sigma \text{ if and only if } aa^{-1}b = bb^{-1}a.
\]

Furthermore, when \( S \) is a totally ordered E-unitary inverse semigroup, Saitô shows that the partial order on \( S \) can be described in terms of the coordinates above. More precisely, if \( S \) is totally ordered then so is \( G = S/\sigma \) and \( a \leq b \) if and only if either \( a\eta < b\eta \) or \( a\eta = b\eta \) and \( aa^{-1} \leq bb^{-1} \). That is, the total order on \( S \) is determined by that on \( G \) and by the totally ordered semilattice \( E \) of idempotents of \( S \). Indeed the total order on \( S \) is (isomorphic to) the lexicographic product of the total order on \( E \) by the total order on \( G \).

Each compatible partial ordering \( \leq \) on an inverse semigroup \( S \) gives rise to three other compatible orderings

\[
x \leq_o y \text{ if and only if } x \geq y
\]

\[
x \leq_i y \text{ if and only if } x^{-1} \leq y^{-1}
\]

\[
x \leq_{oi} y \text{ if and only if } x^{-1} \geq y^{-1}.
\]

When \( S \) is a group or a semilattice, two of these coincide. But when \( S \) is a lattice ordered inverse semigroup which is not a group or a semilattice then all four are distinct. We can regard these orderings as the orbit of the ordering \( \leq \) under the action of the Klein four group on the set of compatible partial orderings on \( S \). We say that two compatible orderings are *conjugate* if they belong to the same orbit of this action.
2 Lexicographic Orderings and Lattice Orderings

Suppose that $S$ is a partially ordered inverse semigroup and that $\theta$ is an order preserving homomorphism of $S$ onto a partially ordered group $G$. Then we shall say that $S$ is lexicographically ordered (over $G$) if and only if, for $a$, $b \in S$,

$$a \theta < b \theta \ implies \ a < b.$$ 

Saitô’s characterization shows that a totally ordered $E$-unitary inverse semigroup is lexicographically ordered over its maximum group homomorphic image. The next result, which is a slight modification of his, shows that lexicographic orderings are maximal orderings on $S$.

**Lemma 2.1** Let $S$ be a partially ordered $E$-unitary inverse semigroup and define a relation $\leq_1$ on $S$ by $a \leq_1 b$ if and only if $a \eta < b \eta$ or $a \eta = b \eta$ and $aa^{-1} \leq bb^{-1}$. Then $\leq_1$ is a compatible lexicographic partial order on $S$ which extends $\leq$. Conversely, if $S$ is lexicographically ordered over $G$ by $\leq$ then $a \leq_1 b$ if and only if $a \eta < b \eta$ or $a \eta = b \eta$ and $aa^{-1} = bb^{-1}$; that is, $\leq = \leq_1$.

**Proof.** It is straightforward to show that $\leq_1$ is a compatible partial ordering on $S$ and that $S$ is lexicographically ordered over $G$. Suppose now that $a \leq b$. Then, since $\eta$ is order preserving, either $a \eta < b \eta$ or $a \eta = b \eta$. In the latter case, since $S$ is $E$-unitary, we have $a = aa^{-1}a \leq aa^{-1}b = bb^{-1}a$ so that $aa^{-1} \leq bb^{-1}aa^{-1}$. Similarly, $b = bb^{-1}b \geq bb^{-1}a = aa^{-1}b$ so that $bb^{-1} \geq aa^{-1}bb^{-1}$. Since idempotents in $S$ commute, it follows that $aa^{-1} \leq bb^{-1}aa^{-1} = aa^{-1}bb^{-1} \leq bb^{-1}$. That is, $a \leq b$ implies $a \leq_1 b$ so that $\leq_1$ extends $\leq$.

Conversely, suppose that $S$ is lexicographically ordered over $G$ by $\leq$, and suppose that $a \leq_1 b$. Then either $a \eta < b \eta$, in which case, since $S$ is lexicographically ordered, $a < b$, or $a \eta = b \eta$ and $aa^{-1} \leq bb^{-1}$. But the latter implies

$$a = aa^{-1}a \leq bb^{-1}a = aa^{-1}b \leq bb^{-1}b = b$$

since $(a, b) \in \sigma$ if and only if $aa^{-1}b = bb^{-1}a$. Thus $a \leq b$. Hence $\leq$ extends $\leq_1$ and so, since $\leq_1$ extends $\leq$ by the first paragraph, $\leq_1$ coincides with $\leq$.\[\square\]
Corollary 2.2 Let $S$ be an $E$-unitary inverse semigroup with maximum group homomorphic image $G$. Then every maximal compatible partial ordering on $S$ is lexicographically ordered over $G$.

When the inverse semigroup in question is semilattice ordered, we can say something stronger. Before stating the results, we shall need the following lemma.

Lemma 2.3 Let $S$ be a $\lor$-semilatticed [$\land$-semilatticed] inverse semigroup. Then $E(S)$ is a $\lor$-subsemilattice [$\land$-subsemilattice] of $S$.

Proof. Let $e, f$ be idempotents of $S$. Then $e \lor f$ commutes with $e$ and $f$. Hence, since $e, f$ are idempotents, $(e \lor f)^{-1}$ commutes with $e, f$. Thus, since $S$ is a $\lor$-semilatticed semigroup, $e \lor f$ also commutes with $(e \lor f)^{-1}$ and so $e \lor f$ belongs to a subgroup of $S$. But $(e \lor f)^2 = (e \lor f)^2$ which, in a group, implies that $e \lor f$ is the identity and thus an idempotent. $

Lemma 2.4 Let $S$ be an $E$-unitary inverse semigroup and suppose that $(a, b) \in \sigma$. Then $a \leq b$ if and only if $aa^{-1} \leq bb^{-1}$ or, equivalently, $a^{-1}a \leq b^{-1}b$. If, further, $S$ is $\lor$-semilatticed then $aa^{-1} \lor bb^{-1} = (a \lor b)(a \lor b)^{-1}$ and $a^{-1}a \lor b^{-1}b = (a \lor b)^{-1}(a \lor b)$. While if $S$ is $\land$-semilatticed $aa^{-1} \land bb^{-1} = (a \land b)(a \land b)^{-1}$ and $a^{-1}a \land b^{-1}b = (a \land b)^{-1}(a \land b)$.

Proof. The first statement in the lemma follows exactly as in the proof of Lemma 2.1 so suppose that $S$ is $\lor$-semilatticed. Then, from Lemma 2.3, the idempotents form a $\lor$-subsemilattice of $S$.

Since $aa^{-1} \lor bb^{-1} \geq aa^{-1}$, and each side of the inequality is idempotent, it follows that
\[
aa^{-1} \lor bb^{-1} \geq (aa^{-1} \lor bb^{-1})aa^{-1} \\
= (a \lor bb^{-1}a)a^{-1} \\
= (a \lor aa^{-1}b)a^{-1} \text{ since } (a, b) \in \sigma \\
\geq aa^{-1}ba^{-1} = ba^{-1}
\]
since $S$ E-unitary implies $ba^{-1}$ is idempotent when $(a, b) \in \sigma$. Similarly, $aa^{-1} \lor bb^{-1} \geq ab^{-1}$.

Now we have
\[
a(a \lor b)^{-1} = [(a \lor b)a^{-1}]^{-1} = [aa^{-1} \lor bb^{-1}]^{-1} = aa^{-1} \lor ba^{-1}
\]
since \((a, b) \in \sigma\) implies \(ba^{-1}\) idempotent and idempotents form a subsemilattice of \(S\); Lemma 2.4. Similarly,

\[
b(a \lor b)^{-1} = bb^{-1} \lor ab^{-1}
\]

and so

\[
(a \lor b)(a \lor b)^{-1} = a(a \lor b)^{-1} \lor b(a \lor b)^{-1} = (aa^{-1} \lor ba^{-1}) \lor (bb^{-1} \lor ab^{-1}) = (aa^{-1} \lor bb^{-1}) \lor (ab^{-1} \lor ba^{-1}) = aa^{-1} \lor bb^{-1}
\]

by the second paragraph.

The results concerning meets follow dually. 

In Saitô’s construction, the group \(G = S/\sigma\) is a totally ordered group. The next result shows that this is inevitable if we hope to construct a lattice ordered inverse semigroup using a lexicographic ordering.

**Proposition 2.5** Let \(S\) be a lattice ordered inverse semigroup and let \(\theta\) be a lattice semigroup homomorphism of \(S\) onto a lattice ordered group \(G\). If \(S\) is lexicographically ordered over \(G\) then \(S\) is a group or \(G\) is totally ordered.

**Proof.** Let \(1\) denote the identity of \(G\) and set \(U = \{x \in S : x\theta = 1\}\). If \(G\) is not totally ordered there exist \(c, d \in S\), \(c, d \notin U\) such that \(c\theta \lor d\theta = 1\); see Lemma 1.4. Let \(v = c \lor d\). Then, for \(x \in U\), \(c\theta < x\theta\) and so, since \(S\) is lexicographically ordered, \(c < x\). Likewise \(d < x\) and so \(c \lor d \leq x\). But, since \(\theta\) is a lattice homomorphism and \(c\theta \lor d\theta = 1\), \(c \lor d \in U\). Hence \(v\) is the least element of \(U\).

Furthermore, \((c^{-1} \land d^{-1})\theta = c^{-1}\theta \land d^{-1}\theta = (c\theta \lor d\theta)^{-1} = 1\) and a similar argument shows that \(u = c^{-1} \land d^{-1}\) is the greatest element of \(U\).

Now \((uc)\theta = c\theta\) and \((ud)\theta = d\theta\) so that \((uc \lor ud)\theta = 1\) whence \(uc \lor ud\) is also the least element of \(U\). That is \(uv = u(c \lor d) = v\). Also \((c^{-1}v)\theta = c^{-1}\theta\) and \((d^{-1}v)\theta = d^{-1}\theta\) and it follows similarly that \(uv = (c^{-1} \land d^{-1})v = u\). Hence \(v = u\) and \(U\) consists of just a single element. Thus \(S\) has a single idempotent and so is a group. 

Suppose now that \(S\) is an E-unitary inverse semigroup and let \(G = S/\sigma\) be its maximal group homomorphic image. Then \(S\) is isomorphic to a \(P\)-semigroup \(P(G, \mathcal{X}, \mathcal{Y})\) where \(\mathcal{X}\) is a down directed partially ordered set,
\( \mathcal{Y} \) is subsemilattice and order ideal of \( \mathcal{X} \) isomorphic to the semilattice of idempotents of \( S \) and \( G \) acts, on the left, on \( \mathcal{X} \), by order automorphisms, in such a way that \( \mathcal{X} = G\mathcal{Y} \). The elements of \( P(G, \mathcal{X}, \mathcal{Y}) \) are the pairs \((a, g) \in \mathcal{Y} \times G \) for which \( g^{-1}a \in \mathcal{Y} \), and multiplication is defined by

\[
(a, g)(b, h) = (a + gb, gh)
\]

where \(+\) denotes greatest lower bound (when it exists) in the partially ordered set \( \mathcal{X} \). We use \(+\) for this operation to avoid confusion with \( \lor \) and \( \land \) when the semigroup in question is a lattice ordered semigroup. Also we shall use \( \leq \) to denote the partial order on the set \( \mathcal{X} \) following our convention of reserving \( \leq \) for the imposed partial order.

If \( S \) is a lattice ordered semigroup then, by Lemma 2.3, the idempotents of \( S \) form a sublattice of \( S \) and so the semilattice of idempotents \( E \) of \( S \) is a lattice ordered semilattice under the operations of join and meet which it inherits from \( S \). Thus, in the \( P\)-semigroup \( P(G, \mathcal{X}, \mathcal{Y}) \), \( \mathcal{Y} \) is a lattice ordered semilattice while \( G \) is a lattice ordered group. If the ordering is lexicographic, then, from Proposition 2.5, \( G \) is a totally ordered group and, from Lemma 2.1, the partial order is given by

\[
(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b
\]

**Lemma 2.6** Suppose that \( S = P(G, \mathcal{X}, \mathcal{Y}) \) is a lattice ordered semigroup. For \( a, b \in \mathcal{Y} \) and for \( g \in G \) such that \( ga, gb \in \mathcal{Y} \),

\[
\begin{align*}
g(a \lor b) & \in \mathcal{Y} \text{ and } g(a \lor b) = ga \lor gb \\
g(a \land b) & \in \mathcal{Y} \text{ and } g(a \land b) = ga \land gb.
\end{align*}
\]

**Proof.** For \( a, b, g \) as above, \((a, g^{-1}), (b, g^{-1})\) belong to \( S \) and are \( \sigma \) related. Hence, by Lemma 2.4, since each \( \sigma \)-class is a sublattice of \( S \),

\[
(a, g^{-1}) \lor (b, g^{-1}) = (c, g^{-1})
\]

where

\[
[(a, g^{-1}) \lor (b, g^{-1})][(a, g^{-1}) \lor (b, g^{-1})]^{-1} = (a, g^{-1})(a, g^{-1})^{-1} \lor (b, g^{-1})(b, g^{-1})^{-1} = (a, 1) \lor (b, 1) = (a \lor b, 1).
\]

Thus \( c = a \lor b \) and so, since \((c, g^{-1}) \in S, gc = g(a \lor b) \in \mathcal{Y} \).
Further, by the dual of Lemma 2.4,

\[(a, g^{-1})^{-1}(a, g^{-1}) \lor (b, g^{-1})^{-1}(b, g^{-1}) = (c, g^{-1})^{-1}(c, g^{-1}).\]

That is

\[(ga, 1) \lor (gb, 1) = (gc, 1) = (g(a \lor b), 1)\]

so that \(ga \lor gb = g(a \lor b)\) as required.

The result involving meets follows in a dual manner.■

**Theorem 2.7** Let \(S = P(G, \mathcal{X}, \mathcal{Y})\) be an E-unitary inverse semigroup where \(\mathcal{Y}\) is a lattice ordered semigroup and where \(G\) acts on \(\mathcal{X}\) in such a way that for \(a, b \in \mathcal{Y}\) and for \(g \in G\) with \(ga, gb \in \mathcal{Y}\),

\[g(a \lor b) \in \mathcal{Y} \text{ and } g(a \lor b) = ga \lor gb\]

\[g(a \land b) \in \mathcal{Y} \text{ and } g(a \land b) = ga \land gb.\]

Suppose further that \(G\) is a totally ordered group. Then under the lexicographic ordering

\[(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b\]

\(S\) is a lattice ordered inverse semigroup.

Conversely, if \(S\) is a lexicographically ordered E-unitary inverse semigroup which is a lattice ordered semigroup, then \(S\) is isomorphic to \(P(G, \mathcal{X}, \mathcal{Y})\) for some \(G, \mathcal{X}, \mathcal{Y}\) as above.

**Proof.** For \((a, g), (b, h) \in S\), set

\[
(a, g) \lor (b, h) = \begin{cases} 
(a, g) & \text{if } g > h \\
(a \lor b, g) & \text{if } g = h \\
(b, h) & \text{if } g < h 
\end{cases}
\]

and define \((a, g) \land (b, h)\) in a dual fashion. Then the conditions linking the action of \(G\) on \(\mathcal{X}\) with the imposed action on \(\mathcal{Y}\) show that \(S\) is closed under the operations of \(\lor\) and \(\land\). Further, it is easy to see that \(S\) is a lattice under \(\leq\) with these joins and meets. Thus, for the direct part of the theorem, it remains to show that multiplication distributes over \(\lor\) and \(\land\). To this end, we show first that \(S\) is a partially ordered semigroup under \(\leq\).
Suppose that \((a, g) \leq (b, h)\) and that \((c, k) \in S\). If \(g < h\) then in the products \((a, g)(c, k), (b, h)(c, k)\) or \((c, k)(a, g), (c, k)(b, h)\) the second coordinates have \(gk < hk\) and \(kg < kh\) respectively, since \(G\) is a partially ordered group. Thus multiplication by \((c, k)\) is compatible in this case. Hence we may suppose that \(g = h\) so that \(a \leq b\).

In the products \((a, g)(c, k)\) and \((b, g)(c, k)\) respectively, the first coordinates are \(a + gc\) and \(b + gc\). But

\[
a + gc = g(g^{-1}a + c)
\]

and, by the linking condition, \(a \leq b\) and \(g^{-1}a, g^{-1}b \in \mathcal{Y}\) imply \(g^{-1}a \leq g^{-1}b\) and so, since \(\mathcal{Y}\) is a partially ordered semigroup under \(\leq\), \(g^{-1}a + c \leq g^{-1}b + c\) and then, by the linking condition again, \(g(g^{-1}a + c) \leq g(g^{-1}b + c)\) so that \(a + gc \leq b + gc\).

In the products \((c, k)(a, g)\) and \((c, k)(b, g)\) the first coordinates are \(c + ka\) and \(c + kb\) respectively. This time \(c + ka = k(k^{-1}c + a)\) where \(k^{-1}c \in \mathcal{Y}\) and so \(k^{-1}c + a \leq k^{-1}c + b\) and then, by the linking condition, \(c + ka = k(k^{-1}c + a) \leq k(k^{-1}c + b) = c + kb\). Hence, again multiplication is compatible.

It follows from this that, in order to prove that multiplication distributes over \(\lor\), we need only consider the case when \((a, g)\) and \((b, h)\) are incomparable under \(\leq\). Thus, from the form of the order, \(g = h\) and, as before, we need only look at first coordinates. Then \((a, g) \lor (b, g) = (a \lor b, g)\) and so the first coordinate of \((c, k)[(a, g) \lor (b, g)]\) is

\[
c + k(a \lor b) = k(k^{-1}c + (a \lor b))
\]

\[
= k[(k^{-1}c + a) \lor (k^{-1}c + b)]
\]

\[
= k(k^{-1}c + a) \lor k(k^{-1}c + b)
\]

\[
= (c + ka) \lor (c + kb)
\]

by the linking condition since \(k^{-1}c + a, k^{-1}c + b \in \mathcal{Y}\) and since \(\mathcal{Y}\) is a lattice ordered semigroup under \(\lor\). Since this is the first coordinate of \([[(c, k)(a, g)] \lor [(c, k)(b, g)]\) it follows that multiplication on the left distributes over \(\lor\). A similar calculation shows that it also distributes over multiplication from the right and so \(S\) is a \(\lor\)-semilatticed semigroup. Likewise, it is a \(\land\)-semilatticed semigroup and so a lattice ordered inverse semigroup.

Conversely, suppose that \(S\) is lexicographically ordered and is a lattice ordered semigroup. Then, by Lemma 2.1, Proposition 2.5, and Lemma 2.6, \(G\) is a totally ordered group which acts on \(\mathcal{X}\) in such a way that the linking conditions hold and the order is given as in the statement of the theorem.
The next result relates the construction in Theorem 2.7 to Saitō’s structure theorem for totally ordered E-unitary inverse semigroups. In such a semigroup, the semilattice of idempotents must be a totally ordered semilattice and Saitō [14] has shown that such a semilattice must be a binary tree. That is, it is a semilattice in which each principal ideal is a chain and in which there do not exist distinct incomparable elements \(a, b, c\) with \(a \land b = b \land c = c \land a\). The result is essentially due to Saitō. Our contribution is to simplify the algebraic structure by using the structure theorem for E-unitary inverse semigroups which was not available to him; [7].

**Corollary 2.8** Let \(G\) be a totally ordered group, \(\mathcal{X}\) a tree and \(\mathcal{Y}\) an ideal of \(\mathcal{X}\) which is a totally ordered binary tree. Suppose, further, that \(G\) acts by order automorphisms on \(\mathcal{X}\) in such a way that \(\mathcal{X} = G\mathcal{Y}\) and if \(a, b \in \mathcal{Y}\), \(g \in G\) with \(ga, gb \in \mathcal{Y}\) then \(ga \leq gb\) if and only if \(a \leq b\). Then, under the lexicographic ordering, \(P(G, \mathcal{X}, \mathcal{Y})\) is a totally ordered E-unitary inverse semigroup. Conversely, any totally ordered E-unitary inverse semigroup is isomorphic to one of this form.

**Proof.** The direct part of the theorem follows immediately from Theorem 2.7.

Conversely, we may suppose that \(S = P(G, \mathcal{X}, \mathcal{Y})\). Since \(\mathcal{Y}\) is a (binary) tree and \(G\) acts on the down directed set \(\mathcal{X}\) in such a way that \(\mathcal{X} = G\mathcal{Y}\), it is easy to see that \(\mathcal{X}\) is a tree. Hence, to complete the proof of the theorem, we need only show that the ordering must be lexicographic.

Suppose, therefore that \(a\eta \prec b\eta\) where \(\eta\) denotes the canonical homomorphism of \(S\) onto \(G\). Since \(\eta\) is order preserving, we cannot have \(a \geq b\) and thus, since \(S\) is totally ordered, \(a < b\). Thus \(S\) is lexicographically ordered.■

Saitō [15] also has shown that the free inverse semigroup on a nonempty set cannot be totally ordered. This follows because the idempotents of non-trivial free inverse semigroups do not form binary trees under the natural partial order. We can see this clearly by considering the free inverse semigroup on one generator as a P-semigroup.

**Example.** Let \(\mathcal{X}\) denote the set of all non-empty integer intervals \([a, b]\) partially ordered by containment; that is

\[
[a, b] \leq [c, d] \text{ if and only if } a \leq c \text{ and } b \geq d.
\]

Then the subset \(\mathcal{Y}\) of intervals which contain 0 is an ideal and subsemilattice of \(\mathcal{X}\). Indeed it is an ideal in the distributive lattice \(\mathcal{X}\). The group \(\mathbb{Z}\) of
integers acts on \( \mathcal{X} \) by order automorphisms

\[
g[a, b] = [g + a, g + b] = g + [a, b].
\]

The P-semigroup \( P = P(\mathbb{Z}, \mathcal{X}, \mathcal{Y}) \) is isomorphic to the free inverse monoid on one generator. Since its semilattice of idempotents is a distributive lattice but not a chain - thus not a binary tree - \( P \) cannot be totally ordered. Likewise, since every non-trivial free inverse monoid contains the free inverse monoid on one generator as a submonoid, it cannot be totally ordered either.

Note however, that the action of \( \mathbb{Z} \) satisfies the linking conditions. For \([a, b], g + [a, b] \in \mathcal{Y}\) if and only if \( a \leq 0, -g \leq b \). While

\[
[a, b] \lor [c, d] = [a \lor c, b \land d] = [a, b] \land [c, d]
\]

Thus the action of \( \mathbb{Z} \) preserves the lattice operations and if \( 0, -g \in [a, b] \land [c, d] \) then \( 0, -g \in [a, b] \lor [c, d] \) and, since \([a, b] \land [c, d]\) is larger, also \( 0, -g \in [a, b] \land [c, d] \).

It follows therefore, from the theorem, that \( P(\mathbb{Z}, \mathcal{X}, \mathcal{Y}) \) is a lattice ordered inverse semigroup under the lexicographic ordering.

The ordering on the idempotents in this example is the natural one. We say that a partially ordered inverse semigroup is naturally ordered if the imposed ordering \( \leq \) extends the natural ordering; that is, \( e = ef = fe \) implies \( e \leq f \). In this case the partial ordering on the idempotents coincides with the natural one. For \( e \leq f \) implies \( e \leq ef \) by the compatibility of multiplication while \( ef \leq e \) in any inverse semigroup so that \( ef \leq e \). Thus \( e = ef \) and the imposed partial ordering coincides with the natural one. Thus a naturally ordered inverse semigroup is lattice ordered, indeed \( \lor \)-semilattice ordered, only if the idempotents form a distributive lattice under the natural partial order. In this case, the linking conditions in the statement of Theorem 2.7 are unnecessary.

**Theorem 2.9** Let \( \mathcal{X} \) be a down directed partially ordered set with \( \mathcal{Y} \) an ideal of \( \mathcal{X} \) which is a distributive lattice. Suppose further that \( G \) is a totally ordered group which acts on \( \mathcal{X} \) by order automorphisms in such a way that \( \mathcal{X} = G \mathcal{Y} \). Then \( P(G, \mathcal{X}, \mathcal{Y}) \) is a naturally ordered E-unitary lattice ordered inverse semigroup under the lexicographic ordering

\[
(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b.
\]
Conversely, if \( S \) is a naturally ordered \( E \)-unitary inverse semigroup which is lattice ordered under the lexicographic ordering then \( S \) is isomorphic to \( P(G, \mathcal{X}, \mathcal{Y}) \) for some \( P(G, \mathcal{X}, \mathcal{Y}) \) as above.

**Proof.** We need only show that the linking conditions of Theorem 2.7 are satisfied. So suppose that \( g \in G, \ a, b \in \mathcal{Y} \) are such that \( ga, gb \in \mathcal{Y} \). Then, since \( G \) acts on \( \mathcal{X} \) by order automorphisms, and the ordering on \( \mathcal{Y} \) is the natural one, \( ga \lor gb \in \mathcal{Y} \) and \( ga \land gb = g(a \lor b) \) and likewise \( ga \land gb = ga + gb \in \mathcal{Y} \) and \( ga \land gb = g(a \land b) \). Thus the linking conditions hold.

Theorem 2.9 applies immediately to the free inverse monoid on a non-empty set \( X \). For, by Scheiblich’s theorem [16], [12], the free inverse monoid on \( X \) is isomorphic to \( P(FG(X), \mathcal{X}, \mathcal{Y}) \) where \( FG(X) \) is the free group on \( X \), \( \mathcal{Y} \) is the set of all finite non-empty sets of reduced words over \( X \cup X^{-1} \) which are closed under initial segments. Thus if \( A \in \mathcal{Y} \) and \( w = x_1x_2\cdots x_n \) is a reduced word which is in \( A \) then \( 1, x_1, x_1x_2, \cdots, x_1x_2\cdots x_{n-1} \) also belong to \( A \). The partial order on \( \mathcal{Y} \) is containment. Thus \( A \leq B \) if and only if \( A \supseteq B \). Since \( \mathcal{Y} \) is evidently closed under unions and intersections, it is a sublattice of the Boolean algebra of all non-empty subsets of \( X \) and is thus a distributive lattice. Hence

**Theorem 2.10** The free inverse semigroup \( FI(X) \) on a non-empty set \( X \) cannot be totally ordered. However, for each total ordering on the free group \( FG(X) \) on \( X \) there is a natural lattice ordering on \( FI(X) \).

**Remark 1.** As in the proof of Theorem 2.9, the linking condition for \( \land \) holds automatically for naturally ordered \( E \)-unitary inverse semigroups. Hence the analog of Theorem 2.9 holds for \( \land \)-semilatticed semigroups.

**Corollary 2.11** Let \( S \) be an \( E \)-unitary inverse semigroup whose maximum group homomorphic image \( G \) can be totally ordered. Then \( S \) admits a natural \( \land \)-semilattice ordering. In particular, this is true if \( G \) is a torsion free abelian group.

### 3 Lattice Orderings on Monogenic Inverse Semigroups.

In this section we turn to consider, in more detail the compatible lattice orderings on the free monogenic inverse monoid. We shall denote this semigroup by \( M \). If the free generator is \( a \) then Gluskin [4] has shown that the
elements of $M$ can be uniquely expressed in the form

$$a^r a^{-s} a^t$$

where $r, s, t$ are non-negative integers with $r, t \leq s$. As pointed out earlier, $M$ can also be represented as the $P$-semigroup $P(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ consists of all finite closed intervals $[m, n]$ of integers and $\mathcal{Y}$ consists of those members $[m, n]$ of $\mathcal{X}$ with $0 \in [m, n]$, and where $\mathbb{Z}$ acts on $\mathcal{X}$ by $g[m, n] = [m + g, n + g]$. The elements of $P(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ consist of the pairs $([m, n], g)$ where $m \leq 0, g \leq n$ and

$$([m, n], g)([r, s], h) = ([m, n] \lor g[r, s], g + h) = ([m \land (g + r), n \lor (g + s)], g + h).$$

In terms of this coordinatization, $a^r a^{-s} a^t$ corresponds to $([r - s, r], r - s + t)$. In particular, $a^r$ corresponds to $([0, r], r)$ while $a^{-s}$ corresponds to $([-r, 0], -r)$ and thus $a^n a^{-n}$ and $a^{-n} a^n$ correspond, respectively, to $([0, r], 0)$ and to $([-r, 0], 0)$.

**Lemma 3.1** Let $M$ be the free monogenic inverse semigroup. Then the relation $\lambda$ on $M$ defined by

$$(u, v) \in \lambda \text{ if and only if } ua^n = va^n \text{ for some } n \geq 0$$

is a lattice ordered semigroup congruence. The quotient $M/\lambda$ is a bicyclic inverse semigroup in which $a\lambda^l (a\lambda^l)^{-1} = 1$.

Dually, the relation $\rho$ defined by

$$(u, v) \in \rho \text{ if and only if } a^{-n} u = a^{-n} v \text{ for some } n \geq 0$$

is a lattice ordered semigroup congruence on $M$. The quotient $M/\rho$ is a bicyclic inverse semigroup in which $(a\rho^i)^{-1} a\rho^i = 1$.

**Proof.** We will find it convenient to use the representation of $M$ as a $P$-semigroup. Let $P_1$ be the $P$-semigroup $P(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^+)$ where $\mathbb{Z}$ acts on itself by $g.m = m - g$. Thus the elements of $P_1$ are the pairs $(m, r)$ with $m \geq -r$ and multiplication is given by

$$(m, r)(x, z) = (m \lor (x - r), r + z).$$

Then it is easy to see that $P_1$ is bicyclic with generators $c = (0, 1)$ and $d = (1, -1)$ with $cd = (0, 0)$ the identity of $P_1$; thus $P_1$ is isomorphic to the bicyclic semigroup $B = \langle a, b : ab = 1 \rangle$. 

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Further, it is easy to see from the multiplication in $M$ and $P_1$ that the map $\phi : M \to P_1$ defined by

$$(m, n, r)\phi = (-m, r)$$

is a homomorphism of $M$ onto $P_1$ and that

$$(m, n, r)\phi = ([x, y], z)\phi \text{ if and only if } m = x, r = z.$$  

We show that this occurs if and only if $([m, n], r)\lambda([x, y], z)$. This guarantees that $\lambda$ is a congruence and that $M/\lambda$ is bicyclic.

Suppose therefore that $m = x, r = z$ and choose $k = (n \lor y) - r$. Then, for $u = ([m, n], r)$ and $v = ([x, y], z)$,

$$(m, n, r)([0, k], k) = ([m, n] \lor [r, r + k], r + k)$$

$$= ([m \land r, n \lor (r + k)], r + k)$$

$$= ([m, r + k], r + k) \text{ since } m \leq r \text{ and } r + k \geq n$$

$$= ([x, z + k], z + k) \text{ since } x = m, z = r$$

$$= ([x, y], z)([0, k], k).$$

Thus, in terms of the Gluskin form for the elements of $M$, $ua^k = va^k$. Conversely, direct calculation shows that $ua^k = va^k$ implies $m = x, r = z$. Thus $\lambda$ is, indeed, the congruence corresponding to $\phi$.

Next, suppose that $(u, v) \in \lambda$ and let $w \in M$. Then $ua^k = va^k$ implies that

$$(u \lor w)a^k = ua^k \lor wa^k = va^k \lor wa^k = (v \lor w)a^k$$

$$(u \land w)a^k = ua^k \land wa^k = va^k \land wa^k = (v \land w)a^k$$

so that $\lambda$ is a latticled semigroup congruence.  

The proof that $\rho$ is also a lattice ordered semigroup congruence follows similarly and is omitted. Note, however, that $\lambda \cap \rho$ is the identity congruence on $M$. Further, from the fundamental homomorphism for lattice ordered semigroups, since $\lambda, \rho \subseteq \sigma$, there exist unique lattice ordered semigroup homomorphisms $\lambda^* : B \to M/\sigma$ and $\rho^* : B \to M/\sigma$ such that $\lambda^*\lambda^* = \eta$ and $\rho^*\rho^* = \eta$. We will use these homomorphisms to prove that the lattice ordering on $M$ is lexicographic. To this end, we will need the following result from [9].
Proposition 3.2 Every lattice ordering on the bicyclic semigroup $B = \langle a, b : ab = 1 >$ is a total ordering. $B$ admits precisely four distinct total orderings. Each ordering is conjugate to the ordering defined as follows:

$b^r a^s \leq b^u a^v$ if and only if $s - r < v - u$ or $s - r = v - u$ and $r \leq s$.

Note that this ordering is lexicographic and its form follows from Saitô’s characterization of total orders on E-unitary inverse semigroups.

Lemma 3.3 Let $\leq$ be a compatible semilattice ordering on the free monogenic inverse monoid $M = \langle a \rangle$. Then either $a > 1$ or $a < 1$.

Proof. Suppose, for example, that $\leq$ is a $\lor$-semilattice ordering. Then it is clear, from distributivity, that $a$ commutes with $a \lor 1$. Direct calculation, using either the Gluskin form of multiplication, or the $P$-semigroup form of multiplication, shows that this implies $a \lor 1 = a^r$ for some $r \geq 0$. Thus, since $\eta$ is a $\lor$-semilattice congruence, $a\eta \lor 0 = r(a\eta)$. Thus $r(a\eta) = 0$, in which case $r = 0$ and $a < 1$, or $r(a\eta) = a\eta$, in which case $r = 1$ and $a > 1$.\]

Theorem 3.4 Every lattice ordering on the free inverse monoid $M = \langle a \rangle$ is a lexicographic ordering. Up to conjugacy, $M$ admits precisely two distinct lattice orderings. Each lattice ordering is conjugate either to the ordering defined by $a^r a^{-s} a^t \leq a^x a^{-y} a^z$ if and only if

$$r - s + t < x - y + z \text{ or } r - s + t = x - y + z \text{ and } r \geq x, t \geq z$$

or to the ordering defined by $a^r a^{-s} a^t \leq a^x a^{-y} a^z$ if and only if

$$r - s + t < x - y + z \text{ or } r - s + t = x - y + z \text{ and } r \leq x, t \leq z.$$

Proof. Let $\leq$ be a compatible lattice ordering on $M$ and suppose that $x\eta > y\eta$ where $x, y \in M$. Then, in the terminology of Lemma 3.1, $(x\lambda^i)\lambda^* > (y\lambda^i)\lambda^*$. But, from Proposition 3.2, every lattice ordering on the bicyclic semigroup is a lexicographic total ordering. Hence $x\lambda > y\lambda$ and so $x\lambda = (x \lor y)\lambda$. Similarly, $x\rho = (x \lor y)\rho$ and so, since $\lambda \cap \rho$ is the identity congruence, $x = x \lor y > y$ since $x\eta > y\eta$ implies $x \neq y$. Thus $\leq$ is a lexicographic ordering.

Next, from Lemma 3.3, we may assume that $a > 1$ where $a$ denotes the generator of $M$. Thus $a\lambda > 1$ and $a\rho > 1$ in $M/\lambda$ and $M/\rho$ respectively. In $M/\lambda$, we have $(aa^{-1})\lambda = 1$. From [9], there are precisely two compatible
total orderings on \( \langle c, d : cd = 1 \rangle \) with \( c > 1 \). They are determined by the relations \( dc > 1 \) and \( dc < 1 \) respectively. Hence there are precisely two distinct total orderings on \( M/\lambda \) with \( a\lambda^1 > 1 \). Dually, there are precisely two distinct total orderings on \( M/\rho \) with \( a\rho^1 > 1 \). Thus there are four distinct lattice orderings, the four intersections, on \( M \) with \( a > 1 \). The four dual orderings in which \( a^{-1}\eta > 1 \) give a total of eight distinct lattice orderings on \( M \). Thus there are two conjugacy classes of lattice orderings.

These classes are determined by \( a > 1 \) and either \( aa^{-1} < 1 \), \( a^{-1}a < 1 \) or \( aa^{-1} > 1 \), \( a^{-1}a < 1 \). The first of these corresponds to the natural partial ordering on the idempotents and is given by the first ordering in the statement of the theorem. The second corresponds to the second ordering in the statement of the theorem.

4 Covering Theorems and Green’s Relations.

Many studies of partially ordered inverse semigroups impose restrictions relating the imposed partial order to the structure of Green’s relations on the semigroup. This is the case with papers of Blyth and McFadden and of the third author. These conditions take the form \( a \leq b \) implies \( a \leq r b \) or implies \( a \leq l b \). They are known as regularity or amenability conditions. Such conditions evidently impose restrictions on the structure of Green’s relations on the semigroups in question, and thus on the structure of the semigroups themselves. For example, it is shown in [8] that a lattice ordered inverse semigroup which admits a natural amenable ordering is necessarily a semilattice of groups. In this section, we use the results of Section 2 to show that without the imposition of amenability, there is essentially no restriction on the Green’s relation structure of naturally ordered inverse semigroups which are lattice ordered or \( \wedge \)-semilattice ordered.

To make this precise, we shall make use of Munn’s characterization of fundamental inverse semigroups [11]. Let \( S \) be an inverse semigroup with semilattice of idempotents \( E \). Then the set of all isomorphisms between principal ideals of \( E \) is an inverse semigroup denoted by \( T_E \). It has semilattice of idempotents isomorphic to \( E \) and is fundamental in the sense that the identity congruence is the only idempotent separating congruence on \( T_E \). Indeed an inverse semigroup with semilattice of idempotents isomorphic to \( E \) is fundamental if and only if it is isomorphic to a full inverse subsemigroup of \( T_E \). If \( \mu \) is the maximum idempotent separating congruence on \( S \) then \( S/\mu \)
is fundamental and is therefore isomorphic to a full inverse subsemigroup of $T_E$. Since Green’s relations, apart from $H$, are preserved and reflected by idempotent separating congruences we can regard the full inverse subsemigroups of $T_E$ as providing a set of invariants for the set of different Green’s relation structures of inverse semigroups.

**Theorem 4.1** Let $S$ be an inverse semigroup whose semilattice of idempotents forms a distributive lattice under the natural partial ordering. Then $S$ is an idempotent separating homomorphic image of a naturally lattice ordered inverse semigroup. Conversely, if an inverse semigroup $S$ is an idempotent separating homomorphic image of a naturally lattice ordered inverse semigroup then the idempotents of $S$ form a distributive lattice under the natural partial ordering.

**Proof.** The inverse semigroup $S$ admits an $E$-unitary inverse cover $P(G, \mathcal{X}, E);$ [7]. Let $F$ be a free group which has $G$ as a homomorphic image and induce an action of $F$ on $\mathcal{X}$ from that of $G$ on $\mathcal{X}$. Then $P(G, \mathcal{X}, E)$ is an idempotent separating homomorphic image of $P(F, \mathcal{X}, E)$ so that $T = P(F, \mathcal{X}, E)$ has $S$ as an idempotent separating homomorphic image. But, from Theorem 2.7, since $E$ is a distributive lattice, $T$ admits a natural lattice ordering.

Conversely, if $S$ is an idempotent separating homomorphic image of a naturally lattice ordered inverse semigroup $T$ then idempotents of $T$ form a distributive lattice under the natural partial ordering. Hence the same is true of the idempotents of $S$.■

**Corollary 4.2** Let $E$ be a distributive lattice and let $T$ be any full inverse subsemigroup of the fundamental inverse semigroup $T_E$. Then there is a lattice ordered inverse semigroup $S$ such that $S/\mu \approx T$.

The same technique applies immediately to semilattice orderings. Firstly, to natural $\wedge$-semilattice orderings by means of Theorem 2.7 and then to $\vee$-semilattice orderings since the order dual of a $\wedge$-semilattice ordered inverse semigroup is a $\vee$-semilattice ordered inverse semigroup.

**Theorem 4.3** Let $E$ be a semilattice and $T$ be a full inverse subsemigroup of $T_E$. Then there is a $\vee$-semilattice ordered [$\wedge$-semilattice ordered] inverse semigroup $S$ such that $S/\mu$ is isomorphic to $T$.
Remark 2. The semigroup $T$ in the statement of Corollary 4.2 need not itself be lattice ordered. Indeed, if $E$ is finite and $T$ is not a semilattice (thus if $T$ is not isomorphic to $E$), it cannot be lattice ordered. For any finite subsemigroup of a lattice ordered inverse semigroup consists entirely of idempotents; [8]

Example 1. Let $E$ be the semigroup generated by the four mappings

$$
a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 5 & 6 & 9 & 5 & 6 & 9 & 9 & 9 \end{pmatrix}
$$

$$
b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 2 & 9 & 7 & 5 & 9 & 7 & 9 & 9 \end{pmatrix}
$$

$$
c = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 9 & 3 & 8 & 9 & 6 & 9 & 8 & 9 \end{pmatrix}
$$

$$
d = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
9 & 7 & 8 & 4 & 9 & 9 & 7 & 8 & 9 \end{pmatrix}
$$

Then $E$ has 9 elements $a, b, c, d, ab, ac, bc, cd$ and $ad = bc$ which is the zero 0 of $E$. If we set $e = a, f = d$ then $E$ becomes a lattice ordered semigroup with $b = e \land f$ and $c = e \lor f$. Further $ad = ef = 0, ac = e \lor ef, cd = f \lor ef, ab = e \land ef, bd = ef \land f$.

The group $\mathbb{Z}$ acts on $E$ by lattice ordered semilattice automorphisms by interchanging $a$ and $d$ and leaving $b, c$ fixed:

$$
\sigma = \begin{pmatrix} a & b & c & d & ab & ac & bd & cd & 0 \\
d & b & c & a & db & dc & da & ca & 0 \end{pmatrix}
$$

Thus the semidirect product $S$ of $E$ by $\mathbb{Z}$ becomes a lattice ordered inverse semigroup under the lexicographic ordering in which the idempotents do not form a distributive lattice - or even a lattice - under the natural partial ordering.

Theorem 4.4 Let $S$ be a lattice ordered $E$-unitary inverse semigroup and suppose that $S$ is lexicographically ordered. Then $S$ is a distributive lattice under the imposed partial order if and only if $E(S)$ is a distributive lattice.

Proof. Since $E(S)$ is a sublattice of $S$, the condition is clearly necessary so it remains to show that it is sufficient. Suppose, therefore, that $E(S)$ is a distributive lattice and that $a \lor b = a \lor c$ and $a \land b = a \land c$ in $S$. Then,
since \( G = \mathbb{S}/\sigma \) is a lattice ordered group, and therefore a distributive lattice, \((b, c) \in \sigma\).

Since \( G \) is totally ordered, one of the three possibilities \( a\sigma^i > b\sigma^i, a\sigma^i < b\sigma^i, a\sigma^i = b\sigma^i \) must hold. Suppose the first. Then, since \( S \) is lexicographically ordered, \( a > b \) and \( a > c \). Then, because \( a \land b = a \land c \), it follows that \( b = c \). A similar conclusion holds if \( a\sigma^i < b\sigma^i \) and it remains to consider the case that \( a\sigma^i = b\sigma^i \).

Because \( S \) is \( E \)-unitary, each of \( ab^{-1}, cb^{-1} \) and \( bb^{-1} \) is an idempotent and \( ab^{-1} \lor bb^{-1} = ab^{-1} \lor cb^{-1} \) while \( ab^{-1} \land bb^{-1} = ab^{-1} \land cb^{-1} \). Thus, since \( E(S) \) is a distributive lattice, \( bb^{-1} = cb^{-1} \) so that \( b = cb^{-1}b \). Similarly, \( c = bc^{-1}c \) and so, since \((b, c) \in \sigma \) implies \( cb^{-1}b = bc^{-1}c \) because \( S \) is \( E \)-unitary, \( b = cb^{-1}b = bc^{-1}c = c \). Thus \( S \) is a distributive lattice under \( \leq \).

To end this paper we shall give necessary and sufficient conditions for a lattice ordered semilattice to be a distributive lattice under the imposed partial order. Our test of distributivity is based on the following well known result; [1].

**Lemma 4.5** A lattice \( L \) is modular if and only if it does not contain a sublattice isomorphic to the lattice \( M \) with the Hasse diagram below:

\[
\begin{array}{c}
M \\
\circlearrowright\text{a} \quad \circlearrowright\text{c} \\
\circlearrowright\text{v} \\
\end{array}
\]

It is distributive if and only if it is modular and does not contain the lattice \( D \) with Hasse diagram

\[
\begin{array}{c}
D \\
\circlearrowleft\text{a} \quad \circlearrowleft\text{b} \quad \circlearrowleft\text{c} \\
\circlearrowright\text{v} \\
\end{array}
\]
We shall need the following simple result.

**Lemma 4.6** Let $E$ be a lattice ordered semilattice. Then, for $a, b \in E$, 
$(a \lor b)(a \land b) = ab$.

**Proof.** Since each element of $E$ is idempotent, $a \lor b = (a \lor b)^2 \geq a \cdot b = ab$ and similarly $a \land b \leq ab$. Thus $(a \lor b)(a \land b) \geq ab(a \land b) = ab$ and $(a \lor b)(a \land b) \leq (a \lor b)ab = ab$ so that $(a \lor b)(a \land b) = ab$.\[\blacksquare\]

**Proposition 4.7** Let $E$ be a semilattice and suppose that $E$ is a lattice
ordered semigroup under the partial order $\leq$. Then $E$ does not contain a sublattice, under $\leq$, isomorphic to $D$.

**Proof.** Suppose that $E$ contains $D = \{a, b, c, u, v\}$. By Lemma 4.6, we then have $uv = ab = ac = bc$. Thus, since $E$ is a commutative idempotent semigroup, $z = uv$ acts as a zero for each member of $D$. Let $d' = a \lor z$, $b' = b \lor z$, and $c' = c \lor z$. Then,

$$
\begin{align*}
    d' \lor b' &= (a \lor z) \lor (b \lor z) = u \lor z = u \lor uv = u \text{ since } u \geq v \\
    d' \land b' &= (a \lor z) \land (b \lor z) = a(a \lor b) \land b(a \lor b) \text{ since } z = ab = ba \\
                   &= (a \land b)(a \lor b) = ab = z \text{ from Lemma 4.6.}
\end{align*}
$$

A similar argument shows that $d' \lor c' = b' \lor c' = u$ and $d' \land c' = b' \land c' = z$ so that $F = \{d', b', c', u, z\}$ is a sublattice of $E$. It is also a subsemigroup of $E$ with identity $u$ since, for example,

$$
ud' = u(a \lor z) = u(a \lor ab) = u(a \lor b)a = (a \lor b)a = a \lor ba = a \lor z = a'.
$$

Further, the partial order on $F$ coincides with the natural partial order. Hence $F$ is a distributive lattice under $\leq$. Therefore $u = z$.

In particular, this means that $ab = a \lor b$ and so

$$
av = a(a \land b) = a \land ab = a \land (a \lor b) = a.
$$

Similar calculations show that $v$ is an identity for $b$ and $c$ and then for $u = a \lor b$.

It follows that $D$ is both a subsemigroup and a sublattice of $E$ and that the ordering on $D$ is the reverse of the natural partial ordering. Hence, under $\leq_D$, $D$ is a lattice ordered semilattice in which the imposed partial ordering coincides with $\leq$. This lattice is therefore distributive and therefore the same is true of $D$ under $\leq$. This is a contradiction. Thus $E$ cannot contain a sublattice isomorphic to $D$.\[\blacksquare\]
Corollary 4.8 Let $E$ be a semilattice and suppose that $E$ is a lattice ordered semigroup under $\leq$. If $E$ is a modular lattice under $\leq$ then $E$ is distributive under $\leq$.

The next example shows that a lattice ordered semilattice need not be a modular lattice, and thus not a distributive lattice, under the imposed partial ordering.

Example 2. Let $U = \{a, b, c, u, v\}$ be a five element lattice isomorphic to the non-modular lattice $M$ and define a semilattice product $\circ$ on $U$ with diagram

\[
\begin{array}{c}
  c \\
  \downarrow \\
  b \\
  \downarrow \\
  u \\
  \diamond \\
  v \\
  \downarrow \\
  a
\end{array}
\]

Then it is straightforward to show that, with this semilattice product and with the lattice ordering inherited from $M$, $U$ is a lattice ordered semilattice. Under the imposed partial ordering $\leq$, $U$ is not a modular lattice.

The semigroup $U$ under $\leq_o$ is clearly another example of a lattice ordered semilattice which is not a modular lattice under the imposed ordering. Clearly, this latter lattice ordered semigroup is isomorphic to that with the product $\circ$ and where the imposed ordering has the same diagram as $M$, but with $b$ and $c$ interchanged. We shall denote this lattice ordered semigroup by $U_o$.

Theorem 4.9 Let $E$ be a semilattice and suppose that $E$ is a lattice ordered semigroup under the partial order $\leq$. Then either $E$ is a distributive lattice under $\leq$ or else $E$ contains a subsemigroup and sublattice isomorphic to $U$ or to $U_o$.

Proof. Suppose that $E$ is not a distributive lattice under $\leq$. Then, by Lemma 4.5 and Proposition 4.7, $E$ contains a sublattice $\{a, b, c, u, v\}$ isomorphic to $M$. Then $c = b \lor c \geq bc \geq b \land c = b$ and so at least one of $c \neq bc$ or $b \neq bc$ must hold; say $c \neq bc$. Then $\{a, c, bc, u, v\}$ is a sublattice of $E$ isomorphic to $M$ in which $c(bc) = bc$. So we may suppose that $E$ contains
a sublattice \( \{a, b, c, u, v\} \) isomorphic to \( M \) in which \( bc = b \) or to one in which \( bc = c \).

Suppose that \( bc = b \). From Lemma 4.6, \( uv = ac = ab \) so that \( z = ab \) acts as a zero for each of \( a, b, c, u, v \). Further \( ub = (a \lor b)b = ab \lor b = z \lor b \) and likewise \( uc = c \lor z \). But, also, \( z \lor c = (a \lor b)c = ac \lor bc = z \lor b \) since \( b = bc \). Thus \( z \lor b = z \lor c = ub = uc \).

In a similar fashion, we find that \( z \land b = z \land c = vb = vc \) so that \( \{z, b, c, ub, vb\} \) is a sublattice of \( E \). Because \( b \neq c \), all five elements must be distinct and so this sublattice is isomorphic to \( M \).

Since \( bc = b \) and \( z \) is a zero for each of \( b, c, u, v \), it follows from the form of the elements of \( V = \{z, ub, vb, b, c\} \), that \( V \) is a subsemigroup of \( E \). Furthermore, as a semilattice, \( V \) is isomorphic to \( U \). Thus, as lattice ordered semigroups, \( V \approx U \).

In the case in which \( bc = c \), a similar analysis shows that \( E \) contains a subsemigroup and sublattice isomorphic to \( U_o \). ■

**Remark 3.** This paper has dealt with lattice ordered inverse semigroups in which the imposed partial ordering was a lexicographic one. This need not be the case. For example, let \( E \) be as in Example 1 above. Then under the Cartesian ordering, \( E \times \mathbb{Z} \) is a lattice ordered inverse semigroup which is not lexicographically ordered. Indeed, the direct product of lattice ordered inverse semigroups is always a lattice ordered semigroup.

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