

# Lattice Ordered Groups and the Structure of Inverse Semigroups

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# Preface

There has long been a symbiotic relationship between the theory of lattice ordered groups and the theory of inverse semigroups. In the first place some of the foundational results in each field are due to A.H. Clifford. Second, Tulane University in New Orleans, where Clifford was a faculty member, became a center for the study of both fields, thanks to Clifford and Paul Conrad who was also on the faculty at Tulane. However this relationship is deeper than the personal relationship between these two mathematical giants. The notes which follow are respectfully dedicated *A.M.D.G.* and to both of them.

Inverse semigroups have a natural ordering which has a deep effect on their structure. This makes them similar to partially ordered groups. Indeed, the category of lattice ordered groups can be regarded, up to categorical equivalence, as a subcategory of the category of inverse semigroups. This correspondence is essentially an almost 50 year old result of Clifford's; cf. Petrich, [10]. Clifford did not explore the categorical relationship and neither shall we when we present his result in Chapter 2.

The purpose of these notes is to explore the relationship between lattice ordered groups and inverse semigroups. The first Chapter provides an introduction to the theory of lattice ordered groups. Like the remainder of the notes, this introduction is by no means complete. It is intended as an introduction only. The aim is to provide enough of a flavor of the theory to provide a basis for exploring ways in which the theory of lattice ordered groups can influence and illuminate the study of the structure of inverse semigroups. A comprehensive up to date account of lattice ordered groups can be found in the book by Darnel, [3].

Chapter 2 provides two examples of the way in which lattice ordered groups can provide insight into the structure of inverse semigroups. The first is the exposition of Clifford's theorem describing the structure of bisimple inverse monoids in terms of their right unit subsemigroup. The second essentially considers the structure of inverse semigroups which are generated by the semigroup of positive elements of a lattice ordered group under the assumption that the structure of the idempotents is closely related to the order structure of the lattice ordered group.

Chapter 3 turns to the study of inverse semigroups which admit a semi-lattice ordering in addition to the natural partial ordering. The first section gives some basic results on lattice ordered semigroups and partially ordered inverse semigroups which are of interest in their own right and useful for the

later sections.

The results of the second section are essentially due to T. Saito who in the middle 1960's carried out a deep analysis of the structure of totally ordered semigroups; in particular of totally ordered inverse semigroups. It is of interest that it was in connection with his investigation of totally ordered inverse semigroups that Saito introduced what are now known as  $E$ -unitary inverse semigroups. The structure theorems for these inverse semigroups permit us to give a simpler account of the structure of totally ordered  $E$ -unitary inverse semigroups than that which was available to Saito. This structure shows in a striking fashion the deep relationship between lattice ordered groups and inverse semigroups which is the focus of these notes.

The final two sections explore in more detail the structure of semilattice ordered inverse semigroups in two situations which are analogous to the two structural examples in Chapter 2. But this time, we deal with them in the opposite order. Section 3 deals with the situation in which the idempotent structure is tied to the order structure in a way which is analogous to what happens in Section 2.3. The final section gives the structure of all semilattice orderings on the bicyclic inverse semigroup.

It should be pointed out that no effort has been made to provide complete coverage of the topics considered in the notes. Many proofs have been omitted. This is the case with highly technical arguments throughout and especially in Sections 3.3 and 3.4. Indeed, essentially no proofs are given in Section 3.4. I just want to point out some aspects of the beautiful relationship that I see between these two fascinating mathematical systems. By the same token no attempt has been made to provide a complete bibliography. Instead, a few source materials have been quoted. As is the case with lattice ordered groups, where Darnel's book provides an extensive bibliography, the reader is referred to the books by Higgins [5], Howie [6], and Petrich [10] for more extensive details of papers on semigroups.

A search of Mathematical Reviews lists 262 papers dated after 1980 involving, in one way or the other, partially ordered semigroups. Few of them seem to be concerned with the topics which are covered here. In this sense this presentation is idiosyncratic but, I hope, interesting despite that. I think there is still much of beauty to explore here.

The notes were prepared for a series of lectures given at the Center of Algebra at the University of Lisbon. The author thanks the Center, and especially Professor G. M. S. Gomes, for their hospitality.

The plan of presentation for the lectures called for five ninety minute lectures on these topics over a two week period. Clearly, there is no way in which all of the material contained here - even though it is sketchy, to say the least - could be covered in detail in such a brief period of time. Only

enough of the main results can be covered to give a flavor of the material. Details, such as they are, will have to be left for the reader - “if any”, to quote my Ph.D. supervisor D. C. J. Burgess who used, mischevously, to slip this phrase into his lectures while dictating at breakneck speed. By providing notes up front, so to speak, I hope, at least, to avoid carpal tunnel strain for those who attend these lectures.

The overall outline of presentation will be:

**Lecture 1:** Introduction to Partially Ordered Groups: Chapter 1;

**Lecture 2:** Lattice Ordered Groups: Chapter 1;

**Lecture 3:** Structural Examples in Inverse Semigroups; Bisimple Inverse Monoids and Inverse Semigroups Separated over a Subsemigroup: Chapter 2;

**Lecture 4:** Introduction to Lattice Ordered Semigroups and Totally Ordered Inverse Semigroups: Chapter 3;

**Lecture 5:** Amenable Orderings on Inverse Semigroups and Semilattice Orderings on the Bicyclic Semigroup: Chapter 3.

# Chapter 1

## Lattice Ordered Groups

### 1.1 Partially Ordered Groups

A semigroup  $S$  is said to be *partially ordered* or a *partially ordered semigroup* if it admits a partial order  $\leq$  which is compatible with left and right multiplication :

$$a \leq b \text{ implies } xay \leq xby \text{ for all } x, y \in S^1.$$

Many naturally occurring semigroups come with an obvious partial ordering. This is the case, for example, for semigroups of subsets of a set. It is also the case with all inverse semigroups. For then the *natural partial order*  $\preceq$  defined by

$$a \preceq b \text{ if and only if } a = eb \text{ for some idempotent } e = e^2 \in S$$

is compatible with multiplication. The partial order can equivalently be defined dually by

$$a \preceq b \text{ if and only if } a = bf \text{ for some } f = f^2 \in S;$$

the relationship between  $e$  and  $f$  is that  $f = b^{-1}eb$ . Nambooripad [5],[6] has shown that a regular semigroup  $S$  is partially ordered by the natural partial order

$$a \preceq b \text{ if and only if } a = eb = bf \text{ for some idempotents } e, f \in S$$

if and only if  $S$  is a locally inverse semigroup. [Because we will be dealing with situations where an inverse semigroup has an imposed partial order as well as the natural one, we will need to distinguish between the two orders.

For this reason, we shall adopt the convention that, when there is a possibility of ambiguity,  $\leq$  denotes the imposed partial order and  $\preceq$  denotes the natural one.]

There are other ways of defining a compatible partial order on a regular semigroup; see [5],[6]. In all of these cases, the partial order is trivial when  $S$  is a group. We shall however be interested in situations when the partial orders on groups are not trivial; indeed when the group is a lattice under the imposed partial order.

Suppose now that  $G$  is a partially ordered group; that is the group  $G$  is partially ordered by  $\leq$ . Then, by the existence of group inverses, compatibility implies that

$$a \leq b \text{ if and only if } 1 \leq a^{-1}b$$

or equivalently

$$a \leq b \text{ if and only if } 1 \leq ba^{-1}.$$

That is, the set  $G^+$  of elements exceeding the identity 1 determines the partial order. This set has the following properties:

- (i)  $G^+$  is a submonoid of  $G$ ;
- (ii)  $aG^+ = G^+a$  for each  $a \in G$ ;
- (iii) 1 is the only invertible element of  $G^+$ .

Condition (i) follows easily from compatibility: if  $a \geq 1$  and  $b \geq 1$  then  $ab \geq a.1 = a \geq 1$ . Condition (ii) is almost immediate since each of the sets  $aG^+$  and  $G^+a$  is easily seen to be  $\{g \in G : g \leq a\}$  while (iii) is a consequence of the following useful observation

$$(iv) \quad a \geq b \text{ if and only if } a^{-1} \leq b^{-1};$$

that is, group inversion is an order anti-isomorphism so that, in particular,  $a \geq 1$  if and only if  $a^{-1} \leq 1$ .

**Proof.** Since  $\geq$  is compatible with multiplication on the right,  $a \geq b$  if and only if  $ab^{-1} \geq 1$ . Then, since  $\geq$  is compatible with multiplication on the left,

$$b^{-1} = a^{-1}.ab^{-1} \geq a^{-1}.1 = a^{-1}.$$

For (iii), suppose that  $a \in G^+$  then  $a \geq 1$  implies  $1 = 1^{-1} \geq a^{-1}$ . Hence  $a^{-1} \in G^+$  if and only if  $a^{-1} = 1$ ; that is,  $a = 1$ .  $\square$

It follows from (iii) that the only compatible partial order on a finite group is the trivial one. Indeed, no non-identity element of  $G^+$  can have

finite order. For otherwise  $G^+$  would have a non-identity invertible element. Thus the study of partially ordered groups inevitably leads us into the infinite domain.

Conversely, suppose that  $G^+$  is a subset of a group  $G$  which has the three properties

- (i)  $G^+$  is a submonoid of  $G$ ;
- (ii)  $aG^+ = G^+a$  for each  $a \in G$ ;
- (iii) 1 is the only invertible element of  $G^+$ .

Then we can define a relation  $\leq$  on  $G$  by setting  $a \leq b$  if and only if  $a^{-1}b \in G^+$ . Since  $aG^+ = G^+a$  for each  $a \in G$ , this definition is equivalent to  $a \leq b$  if and only if  $ba^{-1} \in G^+$ .

**Proposition 1.1.1** *Let  $G$  be a group and let  $G^+$  be a subset of  $G$  which has the properties (i), (ii), (iii). Then  $G$  is a partially ordered group under  $\leq$  and  $G^+ = \{g \in G : g \geq 1\}$ .*

**Proof.** For  $a \in G$ ,  $a^{-1}a = 1 \in G^+$  so that  $a \leq a$ . Next  $a \leq b$  and  $b \leq a$  imply  $a^{-1}b$  and  $b^{-1}a \in G^+$  so that  $a^{-1}b$  is invertible in  $G^+$ . By (iii) this implies that  $a^{-1}b = 1$  so that  $a = b$ . That is,  $\leq$  is antisymmetric. The transitivity of  $\leq$  follows since  $a \leq b$  and  $b \leq c$  implies  $a^{-1}b, b^{-1}c \in G^+$  so that, since  $G^+$  is a submonoid,  $a^{-1}c = a^{-1}b \cdot b^{-1}c \in G^+$ . That is  $a \leq c$ .

Suppose now that  $a \leq b$  and let  $x \in G$ . Then  $(xa)^{-1}xb = a^{-1}x^{-1}xb = a^{-1}b \in G^+$  so that  $xa \leq xb$ . Further  $(ax)^{-1}bx = x^{-1}a^{-1}bx \in x^{-1}G^+x = G^+$  since  $G^+x = xG^+$ . Hence  $\leq$  is compatible with respect to multiplication.

Finally,  $a \geq 1$  if and only if  $a = 1^{-1}a \in G^+$  so that the proof is complete.  $\square$

The submonoid  $G^+$  is called the *cone* of the partial order  $\leq$ . The partial order  $\leq$  determines the cone  $G^+$  and the cone  $G^+$  determines  $\leq$  but, of course,  $G^+$  need not determine the group  $G$  which is being partially ordered.

To see this, suppose that  $G^+$  is the cone of a partial order on a group  $G$ , and let  $H$  be any non-trivial group. Then we can define a partial order on  $G \times H$  by

$$(g_1, h_1) \leq (g_2, h_2) \text{ if and only if } g_1 \leq g_2 \text{ and } h_1 = h_2.$$

It is easy to see that the cone of this partial order is  $\{(g, 1) : g \in G^+\} = G^+ \times \{1\}$  which is also the cone of the partially ordered subgroup  $G \times \{1\}$ . It is natural, therefore, to want to characterize the subgroup of  $G$  generated by  $G^+$ . Since  $G^+$  is cancellative and reversible, this subgroup is uniquely determined, up to isomorphism, by  $G^+$ . Note that, since  $G^+a = aG^+$  for each  $a \in G^+$ ,  $a, b \in G^+$  implies  $ab \in aG^+ \cap G^+b = G^+a \cap G^+b$  so that  $G^+$  obeys Ore's condition:  $G^+a \cap G^+b \neq \emptyset$  for any  $a, b \in G^+$ .

For the sake of completeness, and because it serves as a model for some later constructions, we outline the construction of the group of quotients of a cancellative semigroup which obeys Ore's condition; the proof is omitted.

First of all, note that there is a free group on any semigroup. That is, given any semigroup  $S$  there is a group  $G(S)$  and a homomorphism  $\eta : S \rightarrow G(S)$  with the following property: given any homomorphism  $\theta : S \rightarrow G$ , with  $G$  a group, there is a unique homomorphism  $\psi : G(S) \rightarrow G$  such that  $\theta = \eta\psi$ . That is, a unique group homomorphism  $G(S) \rightarrow G$  which makes the diagram

$$\begin{array}{ccc} G(S) & & \\ \eta \uparrow & \searrow \psi & \\ S & & G \\ & \nearrow \theta & \end{array}$$

commute. In general, even when  $S$  is cancellative,  $\eta$  is not one-to-one and, in general,  $\eta$  is not onto. However, it is onto if  $S$  regular or finite.

**Theorem 1.1.2** *Let  $S$  be a cancellative monoid in which, for each  $a, b$ ,  $Sa \cap Sb \neq \emptyset$ . Then  $\eta$  is one to one and each element of  $G(S)$  can be written in the form  $a^{-1}b$  with  $a, b \in S$ .*

*On  $S \times S$  the relation  $\sim$  defined by*

$$(a, b) \sim (c, d) \text{ if and only if there exist } u, v \in S \text{ such that } ua = vc, ub = vd$$

*is an equivalence relation on  $S \times S$ . Denote the equivalence class containing  $(a, b)$  by  $[a, b]$ . Then  $S \times S / \sim$  becomes a group under the binary operation*

$$[a, b][c, d] = [ua, vd] \text{ where } ub = vc.$$

*This group is isomorphic to  $G(S)$  with  $\eta$  given by  $b\eta = [1, b]$ . In this case,  $\eta$  is one to one and, if we identify each element of  $S$  with its image under  $\eta$ , then each element of  $G(S)$  has the form  $a^{-1}b$  with  $a, b \in S$ .*

**Corollary 1.1.3** *Let  $S$  be a cancellative submonoid of a group  $G$ . If, for each  $a, b \in S$ ,  $Sa \cap Sb \neq \emptyset$  then each element of the subgroup generated by  $S$  has the form  $a^{-1}b$  with  $a, b \in S$ .*

**Proof.** Let  $\psi$  be the homomorphism  $G(S) \rightarrow G$  induced by the inclusion  $S \subseteq G$ . Then  $G(S)\psi$  is the subgroup generated by  $S$ . Hence, since each element of  $G(S)$  has the form  $a^{-1}b$  with  $a, b \in S$ , the same is true of this subgroup.  $\square$

**Proposition 1.1.4** *Let  $G$  be a partially ordered group. Then  $a \in \langle G^+ \rangle$ , the subgroup generated by  $G^+$  if and only if  $a$  and  $1$  have a common upper bound in  $G$ .*

**Proof.** Suppose that  $a$  and  $1$  have a common upper bound  $x$  in  $G$ . Then  $x \geq 1$  so that  $x \in G^+$  and  $x \geq a$  so  $b = xa^{-1} \in G^+$ . Thus  $a = b^{-1}x \in \langle G^+ \rangle$ .

Conversely, since  $G^+x = xG^+$  for each  $x \in G^+$ ,  $G^+$  obeys Ore's condition and so each element  $a$  of  $\langle G^+ \rangle$  has the form  $b^{-1}c$  with  $b, c \in G^+$ . Then  $a = b^{-1}c \leq 1.c = c$  and  $c \geq 1$  so that  $a$  and  $1$  have a common upper bound in  $G$ .  $\square$

**Theorem 1.1.5** *Let  $G$  be a partially ordered group. Then the following are equivalent:*

- (i)  $G = \langle G^+ \rangle$ ;
- (ii) for each  $a \in G$ ,  $a$  and  $1$  have a common upper (lower) bound in  $G$ ;
- (iii) for each pair  $a, b \in G$ ,  $a$  and  $b$  have a common upper (lower) bound in  $G$ .

**Proof.** The equivalence of (i) and (ii) for upper bounds is immediate from the previous proposition. And it is clear that (iii) implies (ii) for upper bounds so suppose that (ii) holds for upper bounds.

Let  $a, b \in G$ . By (ii),  $a^{-1}b$  and  $1$  have a common upper bound  $x \in G$ ;  $x \geq a^{-1}b, x \geq 1$ . Then  $ax \geq b$  and  $ax \geq a$  so (iii) holds.

Suppose now that (ii) holds for upper bounds and let  $a \in G$ . Then  $a^{-1}$  and  $1$  have an common upper bound  $x \in G$ ;  $x \geq a^{-1}, 1$ . But then  $x^{-1} \leq a$  and  $x^{-1} \leq 1$  because inversion is an order anti-isomorphism. Thus each of (ii) and (iii) is equivalent to its order dual and the result follows.  $\square$

The fact that inversion is an order anti-isomorphism while multiplication is an order isomorphism is very useful when dealing with partially ordered groups. So, to end this section, we investigate it in a little more detail.

Suppose that  $\leq$  is a compatible partial order on a semigroup  $S$ . Then we obtain a new compatible partial order  $\leq_o$  by setting  $a \leq_o b$  if and only if  $a \geq b$ . Of course  $\leq_o$  is just the usual order dual of  $\leq$ . But when  $S$  is an inverse semigroup we also obtain a compatible partial ordering  $\leq_i$  by setting

$$a \leq_i b \text{ if and only if } a^{-1} \leq b^{-1}.$$

We also obtain a compatible partible partial ordering  $\leq_{oi}$  by first forming  $\leq_o$  and then applying inversion so that

$$a \leq_{oi} b \text{ if and only if } b^{-1} \leq a^{-1}.$$

We can interpret these partial orderings in terms of an action of the Klein four group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  on the set of compatible partial orderings on  $S$ . Thus each orbit has 1, 2 or four members; it has one member if and only if the partial ordering is trivial. If  $S$  is a group then  $\leq_{oi}$  coincides with  $\leq$  so that, unless  $\leq$  is trivial, the orbit has two members; thus  $\leq_i$  and  $\leq_o$  also coincide. When  $\leq$  is the natural partial order on an inverse semigroup  $S$  then  $\leq$  coincides with  $\leq_i$  so that, when  $S$  is non trivial the orbit has order two. On the other hand, if  $S$  is inverse  $\leq$  is different from  $\leq_{oi}$  unless the idempotents are trivially ordered. Indeed, in general, when  $S$  is inverse but is not a group or a semilattice each orbit has order four.

## 1.2 Properties of Lattice Ordered Groups

The fact that inversion is an order anti-automorphism while multiplication is an order automorphism means that lattice ordered groups admit a great deal of order symmetry. In particular we have the following useful results.

**Proposition 1.2.1** *Let  $G$  be a partially ordered group and suppose that  $a, b \in G$ . Then  $a$  and  $b$  have a least upper bound  $a \vee b$  in  $G$  if and only if they have a greatest lower bound  $a \wedge b$ ; this occurs if and only if  $a^{-1}$  and  $b^{-1}$  have a least upper bound. Indeed,*

$$\begin{aligned} a \wedge b &= a(a \vee b)^{-1}b & a \vee b &= a(a \wedge b)^{-1}b \\ a \wedge b &= (a^{-1} \vee b^{-1})^{-1} & a \vee b &= (a^{-1} \wedge b^{-1})^{-1}. \end{aligned}$$

Further for every  $g \in G$ ,

$$\begin{aligned} g(a \vee b) &= ga \vee gb & (a \vee b)g &= ag \vee bg \\ g(a \wedge b) &= ga \wedge gb & (a \wedge b)g &= ag \wedge bg. \end{aligned}$$

**Proof.** Suppose that  $a \vee b$  exists. Then  $x \leq a, b$  implies  $ax^{-1}b \geq aa^{-1}b = b$  since  $x \leq a$  implies  $x^{-1} \geq a^{-1}$  and similarly  $ax^{-1}b \geq a$ . Thus  $ax^{-1}b \geq a \vee b$  so that  $x^{-1} \geq a^{-1}(a \vee b)b^{-1}$ ; that is  $x \leq b(a \vee b)^{-1}a$ . On the other hand,  $b(a \vee b)^{-1}a \leq (a \vee b)(a \vee b)^{-1}a = a$  and  $b(a \vee b)^{-1}a \leq b(a \vee b)^{-1}(a \vee b) = b$ . Hence  $b(a \vee b)^{-1}a$  is the greatest lower bound  $a \wedge b$  of  $a$  and  $b$ . Similarly,  $a \wedge b = a(a \vee b)^{-1}b$ .

The other results in the first block follow because inversion is an order anti-isomorphism while those in the second follow from them because multiplication is an order isomorphism.  $\square$

**Corollary 1.2.2** *The following are equivalent for a partially ordered group  $G$ .*

- (i)  $G$  is a  $\vee$ -semilattice under  $\leq$ ;
- (ii)  $G$  is a  $\wedge$ -semilattice under  $\leq$ ;
- (iii)  $a \vee 1$  exists for each  $a \in G$ ;
- (iv)  $a \wedge 1$  exists for each  $a \in G$ ;
- (v)  $a \vee b$  exists for each  $a, b \in G^+$ ;
- (vi)  $a \wedge b$  exists for each  $a, b \in G^+$ ;
- (vii) for each  $a, b \in G^+$  there exists  $c \in G^+$  such that  $G^+a \cap G^+b = G^+c$ .

If  $G$  satisfies one (thus all) of the conditions in the Corollary we say that  $G$  is a *lattice ordered group* or a *lattice group*.

**Theorem 1.2.3** *Let  $G$  be a lattice ordered group under a partial order  $\leq$  then  $G$  is a distributive lattice under  $\leq$ .*

**Proof.** To say that  $G$  is a distributive lattice means that, for  $a, b, c \in G$ ,  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  or, equivalently,  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  so that each of the lattice operations distributes over the other. To prove that  $G$  is a distributive lattice we shall use Bergman's characterization: a lattice is distributive if and only if, for  $a, b, c \in G$ ,

$$a \vee b = a \vee c \text{ and } a \wedge b = a \wedge c \text{ together imply } b = c.$$

For, suppose that these two equations hold. Then

$$\begin{aligned} (a \wedge c) &= a(a \vee c)^{-1}c \\ &= a(a \vee b)^{-1}b.b^{-1}c \\ &= (a \wedge b)b^{-1}c \\ &= (a \wedge c)b^{-1}c \end{aligned}$$

whence  $b^{-1}c = 1$  and so  $b = c$ . Hence  $G$  is a distributive lattice.  $\square$

**Example.** Let  $G$  be the group of positive rational numbers under multiplication and let  $G^+$  be the multiplicative semigroup of positive integers under multiplication. Then the corresponding partial order on  $G$  is given by

$$a \leq b \text{ if and only if } b = na \text{ for some positive integer } n.$$

In particular, on  $G^+$  the partial order  $\leq$  is just divisibility. For  $a, b \in G^+$ ,  $a \wedge b = \gcd(a, b)$  while  $a \vee b = \text{lcm}(a, b)$ . The formula

$$a \vee b = a(a \wedge b)^{-1}b$$

then becomes the usual formula

$$\text{lcm}(a, b) \text{gcd}(a, b) = ab.$$

In a similar fashion, let  $H$  be the group of non-zero rational functions over a field  $F$  under multiplication and let  $G^+$  be the set of monic polynomials. Then the corresponding partial order on  $H$  is given by

$$f \leq h \text{ if and only if } h = gf \text{ for some } g \in G^+.$$

The subgroup  $G$  generated by  $G^+$  consists of the rational functions with monic numerator and denominator. It is lattice ordered and on  $G^+$  the equation  $f \wedge h = f(f \vee h)^{-1}h$  becomes  $fh = \text{lcm}(f, h) \text{gcd}(f, h)$ .

It follows from these examples that the usual factorization theorems for integers and polynomials can be interpreted in terms of lattice ordered groups. The primes or irreducible polynomials are orthogonal in the sense of the following definition.

**Definition 1.2.4** *Two elements  $a, b$  of a lattice ordered group  $G$  are said to be orthogonal if  $a \wedge b = 1$ .*

Orthogonal elements are very common in lattice ordered groups and play a crucial role in the theory of such groups. The next proposition lists a number of their basic properties.

**Proposition 1.2.5** *Let  $G$  be a lattice ordered group and let  $a, b, c \in G$ .*

- (i) *if  $a \wedge b = 1$  then  $ab = ba$ ;*
- (ii) *if  $a \wedge b = 1$  and  $c \geq 1$  then  $a \wedge bc = a \wedge c$ ;*
- (iii)  *$(a \vee 1) \wedge (a^{-1} \vee 1) = 1$ ;  $a \vee a^{-1} = (a \vee 1)(a^{-1} \vee 1)$ ;*
- (iv)  *$(a \vee 1)^n = a^n \vee 1$ ;  $(a \wedge 1)^n = a^n \wedge 1$ ;*
- (v)  *$a^n \geq 1$  implies  $a \geq 1$ .*

**Proof.** (i) This follows immediately from the formula  $a \vee b = a(a \wedge b)^{-1}b$  and the fact that the  $\vee$  operation is commutative. For, if  $a \wedge b = 1$  we then have  $ab = a \vee b = b \vee a = ba$ .

(ii) Since  $a \wedge b = 1$ ,

$$\begin{aligned}
a \wedge c &= (a \wedge b)(a \wedge c) \\
&= a(a \wedge c) \wedge b(a \wedge c) \\
&= a(a \wedge c) \wedge ba \wedge bc \\
&= a(a \wedge c) \wedge ab \wedge bc \text{ since orthogonal elements commute} \\
&= a(a \wedge b \wedge c) \wedge bc \\
&= a \wedge bc \text{ since } a \wedge b = 1 \text{ and } c \geq 1 \text{ imply } a \wedge b \wedge c = 1.
\end{aligned}$$

(iii) Suppose that  $1 \leq x \leq (a \vee 1) \wedge (a^{-1} \vee 1)$ . Then  $x \leq a \vee 1$  and  $ax \leq 1 \vee a$  so  $1 \leq x \vee xa \leq a \vee 1$ . But then  $x(a \vee 1) = xa \vee x \leq a \vee 1$  implies  $x \leq 1$ . Hence  $x = 1$  and so  $(a \vee 1) \wedge (a^{-1} \vee 1) = 1$ .

Next  $(a \wedge a^{-1}) \vee 1 = (a \vee 1) \wedge (a^{-1} \vee 1) = 1$  since  $G$  is a distributive lattice under  $\leq$ . Thus  $a \wedge a^{-1} \leq 1$  so that  $1 \leq (a \wedge a^{-1})^{-1} = a \vee a^{-1}$ . Hence, since the join of orthogonal elements is their product,

$$\begin{aligned}
(a \vee 1)(a^{-1} \vee 1) &= (a \vee 1) \vee (a^{-1} \vee 1) \\
&= (a \vee a^{-1}) \vee 1 = a \vee a^{-1}.
\end{aligned}$$

(iv) From (ii), it follows by using induction that if  $a \wedge b = 1$  then  $a^m \wedge b^n = 1$  for all  $m, n \geq 0$ . Hence  $(a \vee 1)^n \wedge (a^{-1} \vee 1)^m = 1$  for each  $a \in G$  and all  $m, n \geq 0$ . Since  $(a \vee 1)^n \geq a^n \vee 1$  and  $(a^{-1} \vee 1)^m \geq a^{-m} \vee 1$ , it follows that

$$1 = (a^n \vee 1) \wedge (a^{-m} \vee 1) = (a^n \wedge a^{-m}) \vee 1$$

so that  $a^n \wedge a^{-m} \leq 1$  whence, by inversion,  $a^{-n} \vee a^m \geq 1$ . If we multiply by  $a^n$  this gives  $a^{m+n} \vee 1 \geq a^n \vee 1$  for all  $m, n \geq 0$ .

We now use induction to prove that  $(a \vee 1)^n = a^n \vee 1$ . The case  $n = 1$  is obvious so suppose that  $(a \vee 1)^n = a^n \vee 1$  and consider

$$\begin{aligned}
(a \vee 1)^{n+1} &= (a \vee 1)^n (a \vee 1) \\
&= (a^n \vee 1)(a \vee 1) \\
&= a^{n+1} \vee a^n \vee a \vee 1 \\
&= (a^{n+1} \vee 1) \vee (a^n \vee 1) \vee (a \vee 1) \\
&= (a^{n+1} \vee 1)
\end{aligned}$$

from the previous paragraph.

(v) Finally, suppose that  $a^n \geq 1$ . Then

$$a^n = a^n \vee 1 = (a^n \vee 1) \vee (a^{n-1} \vee 1) = a^n \vee a^{n-1} \vee 1 = a^n \vee a^{n-1} = a^{n-1}(a \vee 1)$$

since  $a^n \geq 1$ . Thus  $a = a \vee 1 \geq 1$ .  $\square$

**Corollary 1.2.6** *Let  $G$  be a lattice ordered group. Then  $G$  is torsion free ; that is, it has no elements of finite order.*

**Proof.** Suppose  $a^n = 1$  then  $a^n \geq 1$  and  $a^{-n} \geq 1$ . Thus  $a \geq 1$  and  $a^{-1} \geq 1$ , that is  $a \leq 1$ . Hence  $a = 1$ .  $\square$

We shall prove, conversely, that if  $G$  is a torsion free abelian group then  $G$  can be lattice ordered. Indeed  $G$  can be totally ordered. For this, we shall need the following lemma.

**Lemma 1.2.7** *Let  $P$  be a cone in a torsion free abelian group  $G$ . If  $a \in G$ ,  $a \notin P \cup P^{-1}$  then at least one of  $P\{a^n : n \geq 0\}$  or  $P\{a^{-n} : n \geq 0\}$  is a cone on  $G$ .*

**Proof.** Suppose not. Then, since  $P\{a^n : n \geq 0\}$  is not a cone there exists  $1 \neq ha^n \in P\{a^n : n \geq 0\}$  such that  $(ha^n)^{-1} \in P\{a^n : n \geq 0\}$ ; that is  $(ha^n)^{-1} = ka^m$  for some  $k \in P$ ,  $m \geq 0$ . Thus  $a^{-n}h^{-1} = ka^m = a^m k$  and so  $a^{-(m+n)} = kh \in P$ . Similarly, since  $P\{a^{-n} : n \geq 0\}$  is not a cone, there exist  $r, s \geq 0$  so that  $a^{r+s} \in P$ . Hence both  $a^{-(m+n)(r+s)}$  and  $a^{(m+n)(r+s)}$  belong to  $P$  and so, because  $P$  is a cone,  $(m+n)(r+s) = 0$ . Thus at least one of  $m+n, r+s$  is zero. Say  $m+n = 0$ ; then  $m = n = 0$  so  $hk = 1$  and thus, because  $P$  is a cone,  $h = k = 1$ . This contradicts  $ha^n \neq 1$ .  $\square$

**Theorem 1.2.8** *Let  $G$  be an abelian group. Then  $G$  can be lattice ordered (totally ordered) if and only if  $G$  is torsion free.*

**Proof.** It is easy to see that the union of any chain of cones in  $G$  is itself a cone. Hence, by Zorn's Lemma, each cone in  $G$  is contained in a maximal cone  $P$ . If  $P$  is not the cone of a totally ordered group then there exists  $a \in G$  such that  $a \notin P \cup P^{-1}$ . But then Lemma 1.2.7, shows that there is a larger cone which contains (exactly) one of  $a, a^{-1}$ . This contradicts the maximality of  $P$ . Hence  $P$  is the cone of a total order on  $G$ .  $\square$

Proposition 1.2.5 shows that  $a^n \geq 1$  implies  $a \geq 1$  in any lattice ordered group  $G$ . Hence this is true in any in any intersection of lattice orders (or total orders) on  $G$ . For such orders, a strengthened version of Lemma 1.2.7 holds.

**Lemma 1.2.9** *Let  $P$  be a cone in an abelian group such that  $x^n \in P$  implies  $n = 0$  or  $x \in P$ . If  $a \notin P$  then  $P\{a^{-n} : n \geq 0\}$  is a cone which contains  $a^{-1}$  but not  $a$ .*

**Proof.** As before, in order to show that  $P\{a^{-n} : n \geq 0\}$  is a cone, it suffices to prove that  $ha^{-n}$  and  $(ha^{-n})^{-1} \in P\{a^{-n} : n \geq 0\}$  together imply

$ha^{-n} = 1$ . But  $(ha^{-n})^{-1} \in P\{a^{-n} : n \geq 0\}$  implies  $h^{-1}a^n = ka^{-m}$  for some  $k \in P, m \geq 0$ . This gives  $a^{m+n} \in P$  which, since  $a \notin P$  implies  $m+n=0$  whence  $n=0$  and so, since  $P$  is a cone,  $h=1$ . Thus  $P\{a^{-n} : n \geq 0\}$  is a cone in  $G$ .  $\square$

**Theorem 1.2.10** *Let  $G$  be an abelian group. Then a compatible partial order  $\leq$  on  $G$  is the intersection of total orders if and only if  $x^n \geq 1$  implies  $x \geq 1$ .*

**Proof.** We have seen that if  $\leq$  is the intersection of total orders then  $x^n \geq 1$  implies  $x \geq 1$ . So we concentrate on the converse.

Let  $G^+$  be a cone in  $G$  and denote by  $Q$  the intersection of all cones of total orders on  $G$  which contain  $G^+$ . If  $Q \neq G^+$ , let  $a \in Q \setminus G^+$ . By Lemma 1.2.9,  $P = G^+\{a^{-n} : n \geq 0\}$  is a cone containing  $G^+$  and  $a^{-1}$ . By Zorn's Lemma,  $P$  is contained in a maximal cone  $M$  in  $G$ . As in the proof of Theorem 1.2.8,  $M$  is the cone of a total order on  $G$  which contains  $G^+$ . Hence  $a \in M$ , since  $a \in Q$ . But then  $a$  and  $a^{-1}$  are both in  $M$  which implies  $a=1$  and contradicts the fact that  $a \notin G^+$ . Hence  $Q = G^+$ .  $\square$

**Corollary 1.2.11** *Let  $G$  be a lattice ordered abelian group. Then  $\leq$  is the intersection of total orders on  $G^+$ . Thus  $G$  is a subdirect product of totally ordered abelian groups.*

**Proof.** The first assertion is immediate since lattice ordered groups obey  $x^n \geq 1$  implies  $x \geq 1$ .

For the second, let  $\{\leq_i : i \in I\}$  be the set of total orderings on  $G$  which contain  $\leq$  with  $\{G_i : i \in I\}$  the corresponding totally ordered groups. Then the diagonal map  $\theta : G \rightarrow \prod\{G_i : i \in I\}$  defined by

$$x\theta = (\dots, x_i, \dots)$$

with  $x_i = x$  is easily seen to be a group and lattice isomorphism of  $G$  into  $\prod\{G_i : i \in I\}$ .  $\square$

The results which we have just given work nicely for abelian groups because  $P\{a^n : n \geq 0\}$  is automatically a normal subsemigroup for each  $a \in G$  when  $P$  is a cone in  $G$ . If  $G$  is not abelian this is no longer the case. If  $P$  is a cone then, because  $P$  is a normal submonoid of  $G$ ,  $P\{a^n : n \geq 0\}$  is a submonoid but it need not be normal. This means that one has to deal with the submonoid generated by  $P$  and conjugates of  $a$  instead of just by  $P$  and  $a$ . However analogs of the theorems which we have given do hold - though they are more complicated - and permit one to characterize non abelian groups which can be totally ordered. The details can be found in the books of Darnel [3] and Fuchs [4]; see also [1]. In particular,

**Theorem 1.2.12** *Let  $G$  be the free group  $FG(X)$  on a non-empty set  $X$  then  $G$  can be totally ordered.*

The proof of this theorem, which we shall not give, is interesting. It depends on some important properties of central series in free groups.

An interesting sidelight on the last result, which distinguishes the situation for non-abelian lattice ordered groups from the abelian case is the following.

**Theorem 1.2.13** *Let  $G = FG(X)$  be the free group on a non-empty set  $X$ . Then every lattice ordering on  $G$  is a total ordering.*

**Proof.** If  $G$  is not totally ordered then it must contain non-identity elements  $a, b$  with  $a \wedge b = 1$ . But then  $a$  and  $b$  commute. Since  $G$  is a free group, this implies that each of  $a$  and  $b$  is a power of a common element  $c$ ;  $a = c^m$ ,  $b = c^n$  with  $m \geq 1$ . Since  $G$  is a lattice ordered group,  $c^m = a > 1$  implies  $c > 1$  and hence also  $n \geq 1$ . But then  $1 < c \leq a, b$  which contradicts the fact that  $a$  and  $b$  are orthogonal.  $\square$

When we are dealing with non abelian groups, it need no longer be the case that every lattice ordering is an intersection of total orderings. However it can be shown that every lattice ordering is the intersection of right orders.

**Definition 1.2.14** *Let  $G$  be a group a total order  $\leq$  on the set  $G$  is called a right order on  $G$  if  $\leq$  is compatible with right multiplication.*

The arguments shown at the beginning of the course show that every right compatible partial ordering on a group  $G$  is determined by the set  $R^+ = \{g \in G : g \geq 1\}$  just as in the case of two sided compatibility. Thus the cone of a right ordered group is just a submonoid  $P$  of  $G$  such that  $P \cap P^{-1} = \{1\}$  and  $G = P \cup P^{-1}$ .

**Example.** Let  $G$  be the group of  $2 \times 2$  real matrices of the form

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

with  $a > 0$ . Then the subset  $P$  of these matrices with  $a \geq 1$  is the cone of a right ordering which is not left compatible and therefore does not turn  $G$  into a totally ordered group.

If  $P$  is the cone of a right ordered group  $G$  it is immediate that the principal left ideals  $Pa$ ,  $a \in P$ , of  $P$  form a chain under inclusion since  $Pa = \{b \in P : a \leq b\}$ . But the same is also true for principal right ideals. For, let  $a, b \in P$ . Then either  $a^{-1} \leq b^{-1}$  or vice versa since  $P$  is a total order

on the set  $G$ . Suppose, for example,  $a^{-1} \leq b^{-1}$ . Then  $b^{-1}a \geq a^{-1}a = 1$  so that  $b^{-1}a \in P$ . Whence  $a \in bP$ .

We shall end this chapter on lattice ordered groups by describing the congruences on these objects. Before doing this however, for the sake of completeness, we will state Holland's important representation theorem for lattice ordered groups. In order to do this, we need to give a means of constructing non abelian lattice ordered groups.

Let  $X$  be a totally ordered set and let  $G$  be the group of all order isomorphisms of  $X$  onto  $X$ . Then  $G$  becomes a partially ordered group if we set  $f \leq g$  if and only if  $xf \leq xg$  for each  $x \in X$ . For  $f, g \in G$  define  $f \vee g$  by

$$x(f \vee g) = xf \vee xg$$

for each  $x \in X$ . Then  $x \leq y$  implies  $x(f \vee g) \leq y(f \vee g)$  so that  $f \vee g$  is order preserving. Let  $y \in X$ . Then  $y = xf = x'g$  for some  $x, x' \in X$ . Since  $X$  is totally ordered  $x \leq x'$  or  $x' \leq x$ . Suppose, for example, that  $x \leq x'$ . Then  $y = xf \leq x'f$  and  $xg \leq x'g = y$  so that  $y = xf \vee xg$ . Hence  $f \vee g$  is onto.

Suppose now that  $x(f \vee g) = x'(f \vee g)$  where  $x \leq x'$ . Then  $xf \leq x'f$  and  $xg \leq x'g$ . If  $x'f \leq x'g$  then  $xf \leq x'f \leq x'g$  and  $x(f \vee g) = x'(f \vee g)$  implies  $xg = x'g$  whence, since  $g$  is one to one,  $x = x'$ . Similarly,  $x'g \leq x'f$  implies  $x = x'$ . Hence  $f \vee g$  is also one-to-one and so  $f \vee g \in G$ . It follows that  $G$  is a lattice ordered group.  $\square$

**Theorem 1.2.15** *Let  $G$  be a lattice ordered group. Then  $G$  is isomorphic to a subgroup and a sublattice of the group of order isomorphisms of a chain.*

## 1.3 Congruences on Lattice Ordered Groups

**Definition 1.3.1** *A homomorphism  $\theta$  of a lattice ordered group  $G$  into a lattice ordered group  $H$  is an  $l$ -homomorphism if  $\theta$  respects  $\vee$  and  $\wedge$ ; that is,*

$$(a \vee b)\theta = a\theta \vee b\theta \text{ and } (a \wedge b)\theta = a\theta \wedge b\theta$$

for all  $a, b \in G$ .

Since  $a \vee b = (ab^{-1} \vee 1)b$  and  $a \wedge b = (a^{-1} \vee b^{-1})^{-1}$  it is easy to see that a group homomorphism  $\theta$  is an  $l$ -homomorphism if and only if  $(a \vee 1)\theta = a\theta \vee 1$  for each  $a \in G$ .

The kernel  $N$  of an  $l$ -homomorphism is clearly closed under  $\vee$  and  $\wedge$  and, in addition, it is *convex* in the sense that  $a \leq x \leq b$  with  $a, b \in N$  implies  $x \in N$ . The corresponding congruence

$$a \equiv b \text{ if and only if } ab^{-1} \in N \text{ if and only if } a\theta = b\theta$$

is thus a congruence on the lattice of  $G$ . Indeed

**Lemma 1.3.2** *Let  $H$  be a convex  $l$ -subgroup of a lattice ordered group  $G$ . Then the relation  $\equiv$  defined by*

$$a \equiv b \text{ if and only if } ab^{-1} \in H$$

*is a congruence on the lattice  $G$  which is compatible with right multiplication.*

*Conversely, if  $\rho$  is a congruence on the lattice  $G$  which is compatible with right multiplication then the  $\rho$ -class  $H$  containing  $1$  is a convex  $l$ -subgroup of  $G$  and  $(a, b) \in \rho$  if and only if  $ab^{-1} \in H$ .*

**Proof.** Since the proof is straight forward it is omitted.  $\square$

It follows that the convex  $l$ -subgroups of  $G$  form a complete lattice isomorphic to the lattice of right compatible lattice congruences on  $G$ . The latter is a sublattice of the lattice of lattice congruences on  $G$ . It is obvious that the set of right compatible lattice congruences on  $G$  is closed under intersections. That it is closed under join follows because the join of right compatible lattice congruences is the transitive closure of their union. This is their join as lattice congruences. This correspondence is also valid for right compatible congruences on any lattice ordered semigroup.

**Theorem 1.3.3** *Let  $L$  be a lattice. Then the lattice of congruences on  $L$  is distributive.*

**Proof.** Suppose that  $\rho, \sigma, \tau$  are congruences on  $L$ . Then  $\rho \cap (\sigma \vee \tau) \supseteq (\rho \cap \sigma) \vee (\rho \cap \tau)$  so we need only verify the converse. Suppose therefore that  $(a, b) \in \rho \cap (\sigma \vee \tau)$ . Then there exist  $a = u_0, \dots, u_n = b$  such that  $(u_i, u_{i+1}) \in \sigma \cup \tau$ ,  $0 \leq i < n$ . Set  $v_i = (a \wedge b) \vee (u_i \wedge (a \vee b))$  for  $0 \leq i < n$ . Then for  $0 \leq i < n$ ,

$$(v_i, a \vee (u_i \wedge a)) \in \rho \text{ that is } (v_i, a) \in \rho$$

so that  $(v_i, v_{i+1}) \in \rho$ . But also  $(v_i, v_{i+1}) \in \sigma \cup \tau$  since  $(u_i, u_{i+1}) \in \sigma \cup \tau$  and each of  $\sigma, \tau$  is a lattice congruence. Hence  $(v_i, v_{i+1}) \in (\rho \cap \sigma) \cup (\rho \cap \tau)$  and therefore  $(v_0, v_n) \in (\rho \cap \sigma) \vee (\rho \cap \tau)$ . Finally

$$\begin{aligned} v_0 &= (a \wedge b) \vee (a \wedge (a \vee b)) = (a \wedge b) \vee a = a \\ v_n &= (a \wedge b) \vee (b \wedge (a \vee b)) = (a \wedge b) \vee b = b \end{aligned}$$

so that  $(a, b) \in (\rho \cap \sigma) \vee (\rho \cap \tau)$ . Hence  $\rho \cap (\sigma \vee \tau) = (\rho \cap \sigma) \vee (\rho \cap \tau)$  and so the lattice of congruences is distributive.  $\square$

**Corollary 1.3.4** *The lattice of  $l$ -congruences on a lattice ordered group is distributive. The lattice of convex  $l$ -subgroups of a lattice ordered group is distributive.*

**Corollary 1.3.5** *The lattice of right compatible lattice congruences on a lattice ordered semigroup is distributive.*

As an application of this result on congruences we give necessary and sufficient condition for a lattice ordered group to be isomorphic to the direct product of lattice ordered groups.

**Theorem 1.3.6** *Let  $G$  be a lattice ordered group and let  $H$  and  $K$  be convex  $l$ -subgroups of  $G$  such that  $H \cap K = \{1\}$ ,  $HK = G$ . Then*

- (i) *each of  $H, K$  is normal;*
- (ii)  *$G \approx H \times K$ ;*
- (iii)  *$K$  is the unique convex  $l$ -subgroup  $L$  such that  $H \cap L = \{1\}$  and  $HL = G$ .*

**Proof.** (i) Let  $h, k$  be positive elements of  $H, K$  respectively. Then, since each of  $H, K$  is convex,  $h \wedge k \in H \cap K = \{1\}$  so that  $h \wedge k = 1$ . It follows that  $h, k$  commute. Next, since each of  $H, K$  is an  $l$ -subgroup,  $H, K$  are generated by their positive elements and so each element of  $H$  commutes with each element of  $K$  and hence, since  $G = HK$ ,  $H$  and  $K$  are normal subgroups of  $G$ .

(ii) Since  $H$  and  $K$  are normal subgroups, the map  $\theta : G \rightarrow H \times K$  given by  $x\theta = (h, k)$  where  $x = hk$  is a group homomorphism. Further  $hk \geq 1$  implies  $h \geq k^{-1}$  so that  $h \vee 1 \geq k^{-1} \vee 1$ . Whence, since  $H, K$  are convex  $l$ -subgroups,  $k^{-1} \vee 1 \in H \cap K = \{1\}$  so that  $k^{-1} \leq 1$ . That is  $k \geq 1$ . Similarly  $h \geq 1$ . Hence  $x = hk \geq 1$  if and only if  $x\theta = (h, k) \geq (1, 1)$  in  $H \times K$ . Hence  $\theta$  is also an order isomorphism.

(iii) The uniqueness of  $K$  is immediate since the lattice of  $l$ -subgroups is distributive.  $\square$



# Chapter 2

## Constructing Inverse Semigroups

In this chapter we shall give examples of some situations where lattice ordered groups can be used to construct classes of inverse semigroups. It should be pointed out that the structures involved apply in more general settings than those which we will consider. That is, they do not depend in an essential way on the fact that the constructions use lattice ordered groups. However the constructions are neater and more transparent if we assume that we are dealing with lattice ordered groups.

### 2.1 Basic Results on Inverse Semigroups

Recall that Green's relations are defined on a semigroup  $S$  as follows:

$$\begin{aligned} a\mathcal{L}b &\text{ if and only if } S^1a = S^1b \\ a\mathcal{R}b &\text{ if and only if } aS^1 = bS^1 \\ a\mathcal{J}b &\text{ if and only if } S^1aS^1 = S^1bS^1 \\ a\mathcal{H}b &\text{ if and only if } a\mathcal{L}b \text{ and } a\mathcal{R}b \\ a\mathcal{D}b &\text{ if and only if } a\mathcal{L} \circ \mathcal{R}b \end{aligned}$$

where  $S^1$  denotes  $S$  if  $S$  has an identity and  $S^1 = S \cup \{1\}$ , where 1 acts like the identity, otherwise. It is an important result of Green that  $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$  which forces  $\mathcal{D}$  to be an equivalence relation. If  $S$  is a finite semigroup then  $\mathcal{J} = \mathcal{D}$  however, when  $S$  is infinite, this need not be the case.

Each of Green's relations, except  $\mathcal{D}$  defines a quasi-order on  $S$ . For example  $\leq_{\mathcal{L}}$  is defined by

$$a \leq_{\mathcal{L}} b \text{ if and only if } a \in S^1b \text{ or, equivalently, } S^1a \subseteq S^1b.$$

Clearly  $\leq_{\mathcal{L}}$  is compatible with multiplication on the right but it need not be compatible with multiplication on the left. One case in which it is also compatible with multiplication on the left is when the semigroup  $S$  is normal in the sense that  $aS = Sa$  for each  $a \in S$  for then  $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{J} = \mathcal{D}$ .

**Example.** Let  $G$  be a partially ordered group. Then each of Green's relations on  $G^+$  is a compatible partial order which coincides with the opposite of the partial order  $\leq$  on  $G^+$ .

The maximal subgroups of a semigroup  $S$  are precisely the  $\mathcal{H}$ -classes of  $S$  which contain idempotents. An important result of Green's shows that any two maximal subgroups in the same  $\mathcal{D}$ -class of  $S$  are naturally isomorphic. Thus the maximal subgroups belong intrinsically to the  $\mathcal{D}$ -classes of  $S$ .

**Definition 2.1.1** *A semigroup  $S$  is said to be aperiodic if it has no non-trivial subgroups.*

If  $S$  is regular, in the sense below, then  $S$  is aperiodic if and only if  $\mathcal{H}$  is trivial.

**Definition 2.1.2** *A semigroup  $S$  is regular if and only if, for each  $a \in S$  there exists  $x \in S$  such that  $a = axa$ . In this case  $a' = xax$  is a solution to the pair of equations  $a = aya$ ,  $y = yay$ . The element  $a'$  is called an inverse for  $a$  in  $S$  and the semigroup is said to be an inverse semigroup if each element of  $S$  has a unique inverse in  $S$ ; in this case, the unique inverse of  $a$  is denoted by  $a^{-1}$ .*

The next proposition gives some basic properties of regular and inverse semigroups; details can be found in [5], [6] or [10]. We shall develop other results later as we need them.

**Proposition 2.1.3** *Let  $S$  be a semigroup. Then*

- (i)  *$S$  is regular if and only if each  $\mathcal{L}$ -class [each  $\mathcal{R}$ -class] of  $S$  contains an idempotent;*
- (ii)  *$S$  is inverse if and only if each  $\mathcal{L}$ -class [each  $\mathcal{R}$ -class] of  $S$  contains a unique idempotent;*
- (iii)  *$S$  is inverse if and only if it is regular and the idempotents commute;*
- (iv) *if  $S$  is inverse then  $(ab)^{-1} = b^{-1}a^{-1}$  for each  $a, b \in S$ ;*
- (v) *if  $S$  is inverse then  $a^{-1}ea$  is idempotent for each  $e^2 = e, a \in S$ .*

Green's relations on an inverse semigroup take on a very simple form.

**Proposition 2.1.4** *Let  $S$  be an inverse semigroup. Then*

- $a \leq_{\mathcal{L}} b$  if and only if  $aa^{-1} \preceq bb^{-1}$
- $a \leq_{\mathcal{R}} b$  if and only if  $a^{-1}a \preceq b^{-1}b$
- $a \leq_{\mathcal{J}} b$  if and only if there exists  $c \in S$  with  $aa^{-1} = cc^{-1}, c^{-1}c \preceq bb^{-1}$
- $a \leq_{\mathcal{H}} b$  if and only if  $aa^{-1} \preceq bb^{-1}$  and  $a^{-1}a \preceq b^{-1}b$
- $a \mathcal{D} b$  if and only if there exists  $c \in S$  such that  $aa^{-1} = cc^{-1}, c^{-1}c = b^{-1}b$ .

It follows that on inverse semigroups Green's relations are determined by their behavior on the idempotents; this is also true on regular semigroups and, less obviously, on finite semigroups.

**Corollary 2.1.5** *Let  $S$  be an inverse semigroup and let  $e, f$  be idempotents in  $S$ . Then*

- $e \leq_{\mathcal{L}} f$  if and only if  $e \preceq f$
- $e \leq_{\mathcal{R}} f$  if and only if  $e \preceq f$
- $e \leq_{\mathcal{J}} f$  if and only if there exists  $c \in S$  with  $cc^{-1} = e, c^{-1}c \preceq f$
- $e \mathcal{D} f$  if and only if there exists  $c \in S$  such that  $cc^{-1} = e, c^{-1}c = f$ .

Suppose now that  $e$  and  $f$  are commuting idempotents in a semigroup  $S$ . Then  $ef = fe$  is idempotent and  $efS = eS \cap fS$  and  $Se f = Se \cap Sf$ . Hence the sets of principal left and right ideals, or, equivalently, the sets of  $\mathcal{L}$  and  $\mathcal{R}$  classes of an inverse semigroup form semilattices under inclusion. Thus, although inverse semigroups are very different from cancellative semigroups like the cones of lattice ordered groups, they share a pattern in common which we shall explore in this chapter

## 2.2 Bisimple Inverse Monoids

**Definition 2.2.1** *A semigroup  $S$  is bisimple if it consists of a single  $\mathcal{D}$ -class. In this case,  $S$  consists of a single  $\mathcal{J}$ -class as well.*

**Proposition 2.2.2** *Let  $S$  be a bisimple inverse monoid and denote by  $R$  the right unit subsemigroup of  $S$ ; thus  $R = \{a \in S : aa^{-1} = 1\}$ . Then  $R$  is a right cancellative monoid and, for each  $a, b \in R$  there exists  $c \in S$  such that  $Ra \cap Rb = Rc$ . Further, for each  $x \in S$ ,  $Rax \cap Rbx = Rcx$ .*

**Proof.** Let  $a \in Rb$ , where  $a, b \in R$ . Then certainly  $a \in Sb$ . On the other hand,  $a = xb$  with  $x \in S$  implies  $a = ab^{-1}b$ , since  $b = bb^{-1}b$ . But then  $ab^{-1} \cdot (ab^{-1})^{-1} = ab^{-1}ba^{-1} = aa^{-1} = 1$  so that  $ab^{-1} \in R$  and  $a \in Rb$ . That is, for  $a, b \in R$ ,  $a \in Rb$  if and only if  $a \in Sb$ .

Let  $a, b \in R$ . Since  $S$  is bisimple, there exists  $z \in R$  such that  $z\mathcal{L}a^{-1}ab^{-1}b$ ; thus  $z^{-1}z = a^{-1}ab^{-1}b$ . But then  $z = za^{-1}ab^{-1} \cdot b = ub$  where  $u = za^{-1}ab^{-1}$ . Then

$$\begin{aligned} uu^{-1} &= za^{-1}ab^{-1} \cdot ba^{-1}az^{-1} \\ &= za^{-1}ab^{-1}bz^{-1} \text{ since } a^{-1}a, b^{-1}b \text{ are idempotents and so commute} \\ &= z \cdot z^{-1}z \cdot z^{-1} = zz^{-1} = 1 \end{aligned}$$

so that  $u \in R$ . Hence  $z \in Rb$  and similarly  $z \in Ra$ . Conversely, if  $y \in Ra \cap Rb$ , then  $y = ya^{-1}a = yb^{-1}b$  so that  $y = ya^{-1}ab^{-1}b$ . Hence  $y \in Sa^{-1}ab^{-1}b = Sz$ . But, by the first paragraph, since  $y, z \in R$ , this means that  $y \in Rz$ . Hence  $Ra \cap Rb = Rz$ .  $\square$

**Corollary 2.2.3** *Let  $S$  be an aperiodic bisimple inverse monoid and denote by  $R$  the right unit subsemigroup of  $S$ . Then  $R$  is a right  $\vee$ -semilatticed semigroup under*

$$a \leq b \text{ if and only if } Rb \subseteq Ra.$$

There is another way in which the relationship between the cones or lattice ordered groups and the lattice ordered groups themselves is reflected in the relationship between the right unit subsemigroups of bisimple inverse monoids and the structure of bisimple inverse monoids.

First, let us recall the construction of the group of quotients of a cancellative semigroup which we considered in the first section; Theorem 1.2. In the case when  $G^+$  is the cone of a lattice ordered group the construction can be made considerably more precise than in the general setting. For, in this case,  $z = xa = yc$  implies

$$z = u(a \vee c) = u(a \vee c)a^{-1}a = u(c * a)a$$

where  $c * a = (a \vee c)a^{-1} \in G^+$ , so that  $x = u(c * a)$ . Similarly,  $y = u(a * c)$  where  $(a \vee c)c^{-1} = a * c \in G^+$ . Thus

$$\begin{aligned} (a, b) \sim (c, d) & \text{ if and only if } \text{ there exists } u \in G^+ \text{ with } u(c * a)b = u(a * c)d \\ & \text{ if and only if } (c * a)b = (a * c)d \end{aligned}$$

Further, since  $xb = yc$  if and only if  $x = v(c * b), y = v(b * c)$  for some  $v \in G^+$ ,

$$[a, b][c, d] = [v(c * b)a, v(b * c)d] = [(c * b)a, (b * c)d].$$

Of course, this formula permits us to define a binary operation on  $G^+ \times G^+$

$$(a, b)(c, d) = ((c * b)a, (b * c)d).$$

When this is done  $\sim$  is, in fact, a congruence.

This formula for multiplication depends only on the fact that the monoid  $T = G^+$  has the property that, for each  $a, b \in T$ , there exists  $c \vee b \in T$  with  $Tc \cap Tb = T(c \vee b)$ . For then we can define  $c * b \in T$  by  $(c * b)b = c \vee b$ . Of course, unless  $\mathcal{L}$  is trivial,  $c \vee b$  is not uniquely determined and, even given  $c \vee b$ ,  $c * b$  is not uniquely determined unless  $T$  is right cancellative.

However, we have:

**Proposition 2.2.4** *Let  $R$  be a right cancellative semigroup in which*

- (i)  $\mathcal{L}$  is trivial;
- (ii) for each  $b, c \in R$  there exists  $b \vee c \in R$  such that  $Rb \cap Rc = R(b \vee c)$ .

*Then  $R \times R$  is a semigroup under the multiplication*

$$(a, b)(c, d) = ((c * b)a, (b * c)d).$$

*This semigroup  $Q(R)$  is an aperiodic bisimple inverse monoid with right unit subsemigroup isomorphic to  $R$ .*

**Proof.** One can verify directly that  $R \times R$  is a semigroup under the binary operation described above. But it is simpler and cleaner to use a representation of it using one-to-one partial transformations to obtain the result. To this end, for each  $(a, b) \in R \times R$  define  $\rho[a, b] : Ra \rightarrow Rb$  by  $x\rho[a, b] = xb$  for each  $x \in R$ . Since  $R$  is right cancellative this is a one-to-one partial transformation of  $Ra$  onto  $Rb$ . Further  $\rho[a, b]^{-1} = \rho[b, a]$  so that the set of partial maps is closed under taking of inverses.

From the two conditions in the statement of the proposition, given  $b, c \in R$  there is a unique element  $b \vee c \in R$  with  $Rb \cap Rc = R(b \vee c)$ . The domain of  $\rho[a, b]\rho[c, d]$  is

$$(Rb \cap Rc)\rho[a, b]^{-1} = (Rb \cap Rc)\rho[b, a] = R(b \vee c)\rho[b, a] = R(c * b)a$$

and its action is given by

$$\begin{aligned} x(c * b)a\rho[a, b]\rho[c, d] &= x(c * b)b\rho[c, d] \\ &= x(b \vee c)\rho[c, d] \\ &= x(b * c)c\rho[c, d] \\ &= x(b * c)d. \end{aligned}$$

Thus  $\rho[a, b]\rho[c, d] = \rho[(c * b)a, (b * c)d]$  so that the maps form an inverse subsemigroup of the semigroup of one-to-one partial transformations of  $R$ . Since  $\mathcal{L}$  is trivial on  $R$ , the map  $(a, b) \rightarrow \rho[a, b]$  is a bijection from  $R \times R$  onto this set of maps and so  $R \times R$  becomes an inverse semigroup under the multiplication defined in the statement of the proposition.

The idempotents are the pairs  $(a, a)$  with  $a \in R$ . Further, for  $a, b \in R$  we have  $(a, b)(a, b)^{-1} = (a, a)$  and  $(a, b)^{-1}(a, b) = (b, b)$  so that  $(a, a)\mathcal{R}(a, b)\mathcal{L}(b, b)$ . Thus any two idempotents are  $\mathcal{D}$  related. Hence  $Q(R)$  is bisimple and has right unit subsemigroup  $\{(1, a) : a \in R\}$  isomorphic to  $R$ .

Finally, since the idempotents  $(a, a)$  are distinct, it is easy to see that  $\mathcal{H}$  is trivial on  $Q(R)$ .  $\square$

Up to isomorphism,  $Q(R)$  is the unique bisimple inverse monoid with right unit subsemigroup isomorphic to  $R$ . To see this, suppose that  $S$  is a bisimple inverse monoid with right unit subsemigroup  $R$  and let  $x \in S$ . Then, because  $S$  is bisimple, there exists  $b \in R$  such that  $b\mathcal{L}x$ ; that is  $bb^{-1} = 1, b^{-1}b = x^{-1}x$ . Let  $a = bx^{-1}$ . Then  $aa^{-1} = bx^{-1}xb^{-1} = bb^{-1}bb^{-1} = 1$  since  $b^{-1}b = x^{-1}x$ . Thus  $a \in R$  and  $x = xb^{-1}b = a^{-1}b$ . It follows that the map  $Q(R) \rightarrow S$  which sends  $(a, b)$  to  $a^{-1}b$  is onto and it is easy to see that it preserves multiplication and is one-to-one. Hence  $Q(R) \approx S$ .

A similar result holds even if  $R$  is not  $\mathcal{L}$  trivial. Note that since  $R$  is right cancellative,  $a\mathcal{L}b$  if and only if  $a = ub$  for some unit  $u \in R$ . In this more general case the following result holds.

**Theorem 2.2.5** (Clifford [2]) *Let  $R$  be a right cancellative semigroup in which the intersection of principal left ideals is again principal. For each  $b, c \in R$  pick elements  $b \vee c \in R$  such that  $R(b \vee c) = Rb \cap Rc$ ; then there exist unique  $b * c, c * b \in R$  such that  $(b * c)c = b \vee c = (c * b)b$ .*

*The relation  $\sim$  on  $R \times R$  defined by*

$$(a, b) \sim (c, d) \text{ if and only if } c = ua, d = ub \text{ for some unit } u \in R$$

*is an equivalence relation on  $R \times R$ . The set  $Q(R)$  of equivalence classes  $[a, b]$  is a bisimple inverse monoid under the multiplication*

$$[a, b][c, d] = [(c * b)a, (b * c)d].$$

Up to isomorphism,  $Q(R)$  is the unique bisimple inverse monoid with right unit subsemigroup isomorphic to  $R$ .

The right unit subsemigroup of a monoid is automatically right cancellative but it need not be cancellative. The next proposition tells us when the right unit subsemigroup of a bisimple inverse monoid is cancellative.

**Definition 2.2.6** *Let  $S$  be an inverse semigroup. Then  $S$  is  $E$ -unitary if and only if  $e^2 = e = ea$  implies  $a^2 = a$ .*

**Proposition 2.2.7** *Let  $R$  be the right unit subsemigroup of a bisimple inverse monoid  $S$ . Then  $R$  is cancellative if and only if  $S$  is  $E$ -unitary.*

**Proof.** Suppose that  $S$  is  $E$ -unitary and that  $xa = xb$  with  $x, a, b \in R$ . Then, since  $bb^{-1} = 1$ ,  $x = xab^{-1}$  so that  $x^{-1}x = x^{-1}xab^{-1}$  and thus, since  $S$  is  $E$ -unitary,  $ab^{-1}$ , is idempotent. This, in turn, implies that  $b^{-1}.ab^{-1}.b$  is idempotent and so, again because  $S$  is  $E$ -unitary,  $b^{-1}a$  is idempotent. Thus

$$1 = bb^{-1}.aa^{-1} = b.b^{-1}ab^{-1}a.a^{-1} = ab^{-1}$$

whence  $b = ab^{-1}b \preceq a$  in the natural partial order on  $S$ . Dually  $a \preceq b$  so that  $a = b$ .

Conversely, if  $R$  is cancellative, suppose that  $ex = e = e^2$  in  $S$ . Then, because  $S$  is bisimple,  $x = a^{-1}b$  and  $e = c^{-1}c$  for some  $a, b, c \in R$ . Then  $1 = cec^{-1} = ca^{-1}bc^{-1}$  so that  $c^{-1} = c^{-1}ca^{-1}bc^{-1} = a^{-1}bc^{-1}$ . Thus  $ac^{-1} = aa^{-1}bc^{-1} = bc^{-1}$  which gives  $ae = be$ . Since idempotents commute, this gives  $aea^{-1}a = beb^{-1}b$  and so  $fa = fb$  for  $f = aea^{-1}beb^{-1}$ . But  $f = d^{-1}d$  for some  $d \in R$  so we have  $d^{-1}da = d^{-1}db$  and thus  $da = db$ . But  $R$  is cancellative whence  $a = b$  and so  $x = a^{-1}b = a^{-1}a$  is idempotent. That is,  $S$  is  $E$ -unitary.  $\square$

**Corollary 2.2.8** *Let  $R$  be a monoid then  $R$  is the right unit subsemigroup of an aperiodic  $E$ -unitary bisimple inverse monoid if and only if  $R$  is the cone  $G^+$  of a right  $\vee$ -semilattice ordered group  $G$ . In this case  $Q(R) = \{(a, b) : a, b \in G^+\}$  under the binary operation*

$$(a, b)(c, d) = ((c \vee b)b^{-1}a, (b \vee c)c^{-1}d)$$

where the multiplication is carried out in  $G$ .

As a special case of this construction, we have the usual representation of the bicyclic semigroup  $B$  as a semigroup of pairs  $(a, b)$  of integers with  $a, b \geq 0$  and binary operation

$$(a, b)(c, d) = ((c \vee b) - b + a, (b \vee c) - c + d)$$

which arises from taking  $R$  to be the non-negative integers and  $G = \mathbb{Z}$ .

## 2.3 Inverse Semigroups Separated over a Subsemigroup

If  $R$  is the right unit subsemigroup of a bisimple inverse monoid  $S$  then, as we saw in the last section, the idempotents of  $S$  have the form  $a^{-1}a$  with  $a \in R$ . Further, for  $a, b \in R$ ,

$$Sa \cap Sb = Sa^{-1}a \cap Sb^{-1}b = Sc^{-1}c = Sc$$

for some  $c \in R$ . In the case when  $R$  is actually the positive cone of a lattice ordered group, this translates to the condition

$$a^{-1}ab^{-1}b = (a \vee b)^{-1}(a \vee b).$$

In this section, we consider inverse semigroups generated by the positive cone  $S$  of a lattice ordered group  $G$  subject to the requirement that, for all  $a, b \in S$ ,

$$a^{-1}ab^{-1}b = (a \vee b)^{-1}(a \vee b) \text{ and } aa^{-1}bb^{-1} = (a \vee b)(a \vee b)^{-1}.$$

**Definition 2.3.1** *Let  $S$  be the positive cone of a lattice ordered group  $G$ . Then an inverse monoid  $T$  is said to be separated over  $S$  if it is generated by  $S$  as an inverse semigroup and for all  $a, b \in S$ ,*

$$a^{-1}ab^{-1}b = (a \vee b)^{-1}(a \vee b) \text{ and } aa^{-1}bb^{-1} = (a \vee b)(a \vee b)^{-1}.$$

If  $S$  is actually the positive cone of a totally ordered group then it is easy to see that any inverse semigroup generated by  $S$  is actually separated over  $S$ .

Let  $S$  be any inverse semigroup. Then there is a free inverse semigroup  $I(S)$  over  $S$ . This semigroup has the following properties:

there is a homomorphism  $\eta : S \rightarrow I(S)$  such that, for each homomorphism  $\theta : S \rightarrow T$  of  $S$  into an inverse semigroup  $T$ , there is a unique homomorphism  $\phi : I(S) \rightarrow T$  (of inverse semigroups) which makes the diagram

$$\begin{array}{ccc} I(S) & & \\ \eta \uparrow & \searrow \phi & \\ S & & T \\ & \nearrow \theta & \end{array}$$

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commute. The existence of  $I(S)$  follows from standard categorical arguments. But perhaps the easiest way to see the existence of  $I(S)$  is to make use of the free inverse semigroup  $FI(S)$  on the set  $S$ .

For let  $F$  denote the free semigroup on  $S$ . Then  $F$  can be regarded as the subsemigroup of  $FI(S)$  generated by the elements  $\{s : s \in S\}$ . Thus there is a homomorphism  $\zeta : F \rightarrow S$ .  $I(S)$  is the quotient of  $FI(S)$  modulo the congruence  $\pi$  on  $FI(S)$  generated by  $\zeta \circ \zeta^{-1}$ . Note that, since  $\zeta \circ \zeta^{-1} \subseteq \pi \cap (F \times F)$ , there is a homomorphism  $\eta : S \rightarrow I(S)$  such that  $\zeta\eta = \pi^\sharp$  where  $\pi^\sharp$  is the natural homomorphism  $F(S) \rightarrow F(S)/\pi = I(S)$ .

The inverse semigroup  $E(S)$  is the quotient of  $I(S)$  subject to the relations

$$a^{-1}ab^{-1}b = (a \vee b)^{-1}(a \vee b) \text{ and } aa^{-1}bb^{-1} = (a \vee b)(a \vee b)^{-1}$$

for all  $a, b \in S$ . It is the free inverse semigroup separated over  $S$ . Thus if  $S$  is the positive cone of a totally ordered group — indeed, if the principal left and right ideals of  $S$  form a chain under inclusion —  $I(S) = E(S)$ .

**Lemma 2.3.2** *Let  $S$  be the positive cone of a lattice ordered group. Then each element of  $E(S)$  can be written in the form  $ab^{-1}c$  with  $a, b, c \in S$  and  $b \geq a \vee c$ .*

**Proof.** First, for  $a \in S$ ,  $a = aa^{-1}a$ ,  $a^{-1}$  each have the appropriate form. So suppose that  $b \geq a \vee c$  and consider

$$\begin{aligned} x.ab^{-1}c &= (xa).(xa)^{-1}(xa).b^{-1}b.b^{-1}c \\ &= (xa)(xa \vee b)^{-1}(xa \vee b).b^{-1}c \end{aligned}$$

for  $x \in S$ . Now  $xa \vee b = ub$  with  $u \in S$  so

$$\begin{aligned} x.ab^{-1}c &= xa.b^{-1}u^{-1}ub.b^{-1}c \\ &= xa.b^{-1}u^{-1}uc \text{ since idempotents commute} \\ &= xa(ub)^{-1}uc. \end{aligned}$$

Further  $ub = xa \vee b \geq xa$  and, since  $b \geq c$ ,  $ub \geq uc$ . Hence  $x.ab^{-1}c$  has the correct form for each  $x \in S$ .

Next

$$\begin{aligned} x^{-1}.ab^{-1}c &= x^{-1}aa^{-1}u^{-1}c \text{ where } b = ua \text{ with } u \in S \text{ since } b \geq a \\ &= x^{-1}.xx^{-1}.aa^{-1}.u^{-1}c \\ &= x^{-1}(x \vee a)(x \vee a)^{-1}.u^{-1}c \\ &= x^{-1}xvv^{-1}x^{-1}u^{-1}c \text{ where } x \vee a = xv \\ &= vv^{-1}x^{-1}u^{-1}c \text{ since idempotents commute} \\ &= v(uxv)^{-1}c. \end{aligned}$$

Here  $uxv \geq v$  and  $uxv = u(x \vee a) \geq ua = b \geq c$ . Thus, again the product has the required form.

Similarly the set of words of the form  $ab^{-1}c$  with  $b \geq a \vee c$  is closed under multiplication on the right by elements of  $S$  and their inverses. Hence each element of  $E(S)$  has the required form.  $\square$

Lemma 2.3.2 gives a coordination for  $E(S)$  but it says nothing about the uniqueness of the coordination. To prove that the coordination is unique and to discover how they the elements multiply we turn to another way in which lattice ordered groups make a natural appearance in the theory of inverse semigroups.

Let  $\mathcal{X}$  be a down directed partially ordered set,  $\mathcal{Y}$  an ideal and subsemilattice of  $\mathcal{X}$  and let  $G$  be a group which acts by order automorphisms of  $\mathcal{X}$  on the left in such a way that  $\mathcal{X} = G.\mathcal{Y}$ . Then the set  $P(G, \mathcal{X}, \mathcal{Y}) = \{(a, g) \in \mathcal{Y} \times G : g^{-1}.a \in \mathcal{Y}\}$  is an  $E$ -unitary inverse semigroup under the multiplication

$$(a, g)(b, h) = (a \wedge gb, gh).$$

Conversely, every  $E$ -unitary inverse semigroup is isomorphic to one of this form.

Semigroups constructed in this way are called  $P$ -semigroups .

**Lemma 2.3.3** *Let  $P(G, \mathcal{X}, \mathcal{Y})$  be a  $P$ -semigroup. Then*

- (i)  $(a, g)^{-1} = (g^{-1}a, g^{-1})$ ;
- (ii)  $(a, g)(a, g)^{-1} = (a, 1)$  and  $(a, g)^{-1}(a, g) = (g^{-1}a, 1)$ ;
- (iii)  $P(G, \mathcal{X}, \mathcal{Y})$  has semilattice of idempotents isomorphic to  $\mathcal{Y}$ .

This lemma allows Green's relations to be calculated easily on  $P(G, \mathcal{X}, \mathcal{Y})$ . In particular

$$(a, 1)\mathcal{D}(b, 1) \text{ if and only if there exists } g \in G \text{ such that } b = ga.$$

Thus  $P(G, \mathcal{X}, \mathcal{Y})$  is bisimple if and only if  $G$  acts transitively on  $\mathcal{X}$ .

**Example.** Let  $G^+$  be the positive cone of a lattice ordered group  $G$ . Then  $G$  acts on itself on the left by multiplication:  $g.a = ag^{-1}$ . Let  $\mathcal{X} = G$ ,  $\mathcal{Y} = G^+$ . Then, under the reverse of the usual ordering,  $\mathcal{X}$  is a (down) directed partially ordered set with  $\mathcal{Y}$  as an order ideal and subsemilattice. Then  $(a, g) \in P$  if and only if  $a, ag = g^{-1}.a \in G^+$  if and only if  $a \in G^+$  and  $g = a^{-1}b$  with  $b \in S$ . This semigroup is a bisimple inverse monoid with right unit subsemigroup  $\{(1, b) : b \in S\} \approx S$ .

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If  $G = \mathbb{Z}$  is the group of integers (under addition) then the semigroup  $P$  obtained in this way is just the bicyclic semigroup  $B = \langle a, b : ab = 1 \rangle$  with  $a = (0, 1)$  and  $b = (1, -1)$ .

There are lots of other ways in which we can use the lattice ordered group  $G$  to construct  $P$ -semigroups.

**Example.** Let  $S = G^+$  be the positive cone of a lattice ordered group  $G$ . The  $G$  acts on the set  $\mathcal{X}$  of all intervals  $[a, b] = \{x \in G : a \leq x \leq b\}$  by  $g \cdot [a, b] = [ga, gb]$ .  $\mathcal{X}$  is a down directed partially ordered set under

$$[a, b] \leq [c, d] \text{ if and only if } [a, b] \supseteq [c, d].$$

The set  $\mathcal{Y}$  of intervals  $[a, b]$  with  $a \leq 1 \leq b$  is an order ideal and subsemilattice with

$$[a, b] \wedge [c, d] = [a \wedge b, c \vee d].$$

$P(G, \mathcal{X}, \mathcal{Y})$  consists of the pairs  $([u, v], g)$  such that

$$\begin{array}{ll} (i) & u \leq 1 \leq v \\ (ii) & g^{-1}u \leq 1 \leq g^{-1}v \end{array}$$

that is  $u \leq 1, g \leq v$ . It is easily seen that the map  $\eta : S \rightarrow P(G, \mathcal{X}, \mathcal{Y})$  given by  $a\eta = ([1, a], a)$  is one-to-one and, since

$$([1, a], a)([1, b], b) = ([1, a] \wedge a[1, b], ab) = ([1, a] \wedge [a, ab], ab) = ([1, ab], ab)$$

a homomorphism of  $S$  into  $P(G, \mathcal{X}, \mathcal{Y})$ .

**Theorem 2.3.4** *With the notation above,  $P(G, \mathcal{X}, \mathcal{Y}) \approx E(S)$ .*

**Proof.** First, we show that  $P(G, \mathcal{X}, \mathcal{Y})$  is separated over  $S\eta \approx S$ . For  $a, b \in S$

$$\begin{aligned} (aa^{-1}bb^{-1})\eta &= ([1, a], 1)([1, b], 1) \\ &= ([1, a] \wedge [1, b], 1) = ([1, a \vee b], 1) \\ &= (a \vee b)(a \vee b)^{-1}\eta \end{aligned}$$

and similarly,  $(a^{-1}ab^{-1}b)\eta = (a \vee b)^{-1}(a \vee b)\eta$ .

It follows, from the universal property of  $E(S)$  that  $\eta$  extends to a homomorphism of  $E(S)$  into  $P(G, \mathcal{X}, \mathcal{Y})$ ; we shall also denote this by  $\eta$ . From Lemma 2.3.2, each element of  $E(S)$  has the form  $ab^{-1}c$  with  $b \geq a \vee c$ . Then

$$\begin{aligned} (ab^{-1}c)\eta &= ([1, a], a)([1, b], b)^{-1}([1, c], c) \\ &= ([1, a], a)([b^{-1}, 1], b^{-1})([1, c], c) \\ &= ([1, a] \wedge [ab^{-1}, a] \wedge [ab^{-1}, ab^{-1}c], ab^{-1}c) \\ &= ([ab^{-1}, a], ab^{-1}c) \end{aligned}$$

since  $a \vee c \leq b$  implies  $ab^{-1} \leq 1, ab^{-1}c \leq a$ . Hence  $(ab^{-1}c)\eta = (xy^{-1}z)\eta$  implies  $x = a, ab^{-1} = xy^{-1}, ab^{-1}c = xy^{-1}z$  and so, in turn,  $x = a, y = b, z = c$ . Thus  $\eta$  is one-to-one. To complete the proof, we need only show that  $\eta$  is onto.

Suppose then that  $([u, v], g) \in P(G, \mathcal{X}, \mathcal{Y})$  and set  $a = v, b = u^{-1}v, c = u^{-1}g$ . Then  $u \leq 1$  implies  $u^{-1} \geq 1$  so  $b \in S$  and  $b \geq a$ . Also, since  $u \leq g$ ,  $c = u^{-1}g \geq 1$  thus  $c \in S$  and, since  $v \geq g$ ,  $b = u^{-1}v \geq u^{-1}g = c$ . Hence  $([u, v], g) = (ab^{-1}c)\eta \in E(S)\eta$ .  $\square$

**Corollary 2.3.5** *In  $E(S)$ , each idempotent is of the form  $a^{-1}abb^{-1}$  with  $a, b \in S$ . The idempotents form a distributive lattice anti-isomorphic to  $S \times S$ .*

**Proof.** The idempotents of  $E(S)$  are isomorphic to those of  $P(G, \mathcal{X}, \mathcal{Y})$ . That is they are isomorphic to  $\mathcal{Y} = \{[a^{-1}, b] : a, b \in S\}$  where

$$\begin{aligned} [a^{-1}, b] \leq [c^{-1}, d] & \quad \text{if and only if} \quad [a^{-1}, b] \supseteq [c^{-1}, d] \\ & \quad \text{if and only if} \quad a^{-1} \leq c^{-1}, b \geq d \\ & \quad \text{if and only if} \quad a \geq c, b \geq d. \end{aligned}$$

Hence  $\mathcal{Y} \approx S \times S$  under the reverse of the usual product ordering. Since  $S$  is the positive cone of a lattice ordered group,  $S$  is a distributive lattice and therefore so is  $\mathcal{Y}$ .  $\square$

**Corollary 2.3.6** *The free monogenic (one generator) inverse monoid has a distributive lattice of idempotents.*

**Corollary 2.3.7**  *$E(S)$  is a subdirect product of bisimple inverse monoids.*

**Proof.** Define  $\theta_1 : P(G, \mathcal{X}, \mathcal{Y}) \rightarrow P(G, G, S)$  where  $G$  acts on itself by left multiplication:

$$([a^{-1}, b], g)\theta_1 = (b, g).$$

The codomain semigroup is bisimple with right unit subsemigroup isomorphic to the left-right dual  $S^{op}$  of  $S$ .

Likewise  $\theta_2 : P(G, \mathcal{X}, \mathcal{Y}) \rightarrow P(G, G, S)$  where  $G$  acts on itself by  $g.a = ag^{-1}$  given by

$$([a^{-1}, b], g)\theta_2 = (a, g)$$

is also a homomorphism so that the map  $\theta_1 \times \theta_2$  is an isomorphism of  $P(G, \mathcal{X}, \mathcal{Y})$  onto a subdirect product of the two bisimple inverse monoids.  $\square$

# Chapter 3

## Lattice Ordered Inverse Semigroups

### 3.1 Lattice Ordered Semigroups

In this chapter we return to the study of lattice ordered semigroups. As we have seen, the study of lattice ordered groups inevitably leads to infinite groups. Our first result shows that there is an analogous result for lattice ordered semigroups.

**Proposition 3.1.1** *Let  $S$  be a lattice ordered semigroup. Then every non-trivial subgroup of  $G$  is torsion free.*

**Proof.** Suppose that  $H$  is a finite subgroup of  $S$ ; say  $H = \{h_1, \dots, h_n\}$ . Then  $h = h_1 \vee \dots \vee h_n$  has the property that

$$hh_i = h_1h_i \vee \dots \vee h_nh_i = h$$

since multiplication by  $h_i$  permutes the elements of  $H$ . Likewise  $h_ik = k$  for  $k = h_1 \wedge \dots \wedge h_n$ . But then

$$\begin{aligned} hk &= h(h_1 \wedge \dots \wedge h_n) = hh_1 \wedge \dots \wedge hh_n = h \wedge \dots \wedge h = h \\ &= (h_1 \vee \dots \vee h_n)k = h_1k \vee \dots \vee h_nk = k \vee \dots \vee k = k \end{aligned}$$

so that  $h = k$ . But then, since  $k \leq h_i \leq h$  for  $1 \leq i \leq n$ , we get  $k = h_i = h$  for each  $i = 1, \dots, n$ . Hence  $H$  is trivial.  $\square$

**Corollary 3.1.2** *Any finite lattice ordered semigroup is aperiodic.*

When we consider lattice ordered inverse semigroups we can obtain stronger results. The next lemma is somewhat technical, but useful.

**Lemma 3.1.3** *Let  $S$  be a  $\vee$ -semilatticed inverse semigroup and suppose that  $x^2 = x^3$  for some  $x \in S$ . Then  $x \leq x^2$ .*

**Proof.** Let  $a = x \vee x^2$ . Then  $a^2 = x^2 = a^3 \leq a$ . Now set  $b = a \vee aa^{-1}$ . Then

$$b^2 = a^2 \vee a \vee aa^{-1} \vee a^2 a^{-1} = a \vee aa^{-1} = b$$

since  $a^2 \leq a$  implies  $a^2 a^{-1} \leq aa^{-1}$ . But

$$b = a \vee aa^{-1} = a(a^{-1}a \vee a^{-1}) \in aS$$

and  $a = a^2 \vee a = (a \vee aa^{-1})a = ba \in bS$ . Thus  $a\mathcal{R}b$  and so, since  $b$  is an idempotent,  $b = aa^{-1}$  so  $aa^{-1} \geq a$ . But then  $aa^{-1} \geq a \geq a^2$  which implies  $a^{-1} \geq a^{-1}a \geq a^{-1}a^2 = a^{-3} = a^2$  since  $a^2$  is idempotent. Hence  $a^2 = a^{-2} \geq a^{-1} \geq a^{-1}a^2 = a^2$  and so  $a^{-1} = a^2$  and  $a = a^4 = a^2$ . Thus  $x^2 = x \vee x^2 \geq x$ .  $\square$

**Theorem 3.1.4** *Let  $S$  be a lattice ordered inverse semigroup. Then every finite subsemigroup of  $S$  consists entirely of idempotents.*

**Proof.** Suppose that  $x$  is an element of  $S$  of finite order. Then, by Proposition 3.1.1,  $x^n = x^{n+1}$  for some  $n \geq 1$ . If  $x$  is not idempotent then  $n > 1$  and  $y = x^{n-1} \neq x^n$  so  $y \neq y^2 = y^3 = x^n$ . Thus, from Lemma 3.1.3 and its order dual,  $y \leq y^2 \leq y$ . This contradicts the fact that  $y$  is not an idempotent. Hence  $x$  must be an idempotent.  $\square$

**Corollary 3.1.5** *Let  $S$  be a lattice ordered inverse semigroup. Then  $S$  is  $E$ -reflexive in the sense that  $xy \in E$  implies  $yx \in E$ , where  $E$  is the set of idempotents of  $S$ .*

**Proof.** If  $xy \in E$  then  $(yx)^3 = y(xy)^2x = y(xy)x = (yx)^2$ . Hence  $yx$  is idempotent.  $\square$

**Corollary 3.1.6** *Let  $S$  be a lattice ordered inverse semigroup. Then the semilattice  $E$  of idempotents of  $S$  is a sublattice of  $S$ .*

**Proof.** Let  $e, f \in E$  and set  $x = e \vee f$ . Then  $x^2 = (e \vee f)(e \vee f) = e \vee ef \vee f = x^3$ . Hence  $x$  is idempotent. Similarly  $e \wedge f$  is an idempotent.  $\square$

One can give an alternative proof to this corollary. Indeed, the corresponding result holds for semilatticed inverse semigroups.

**Proposition 3.1.7** *Let  $S$  be a  $\vee$ -semilatticed [ $\wedge$ -semilatticed] inverse semigroup. Then the set  $E$  of idempotents is a subsemilattice of  $S$ .*

**Proof.** Let  $x = e \vee f$ . Then  $ex = e \vee ef = e \vee fe = xe$  and similarly  $xf = fx$ . It follows, by taking inverses, that  $e$  and  $f$  also commute with  $x^{-1}$  and therefore, because multiplication commutes over  $\vee$ , that  $x$  commutes with  $x^{-1}$ . This means that  $x$  belongs to a subgroup of  $S$ . But  $x^2 = x^3$  and so, since  $x$  is in a subgroup,  $x = x^2$ .  $\square$

If  $S$  is any semigroup then it is easy to see that the intersection of cancellative congruences on  $S$  is again a cancellative congruence. Hence there is a least cancellative congruence on any semigroup. This means, in particular, that there is a least group congruence on any inverse semigroup. This congruence  $\sigma$  has been one of the most useful tools in the study in the study of inverse semigroups. It is also useful when we study partially ordered inverse semigroups.

The congruence  $\sigma$  is defined by

$$(a, b) \in \sigma \text{ if and only if } ea = eb \text{ for some } e^2 = e \in S.$$

In terms of the natural partial order  $\preceq$  on  $S$ ,

$$(a, b) \in \sigma \text{ if and only if there exists } c \in S \text{ such that } c \preceq a, b.$$

Thus, from the point of view of the natural partial order, the  $\sigma$ -classes are the connected components of the graph of  $S$  considered as a partially ordered set.

For later use, we collect some results on the relationship between  $\sigma$  and the imposed partial order on a partially ordered inverse semigroup.

**Theorem 3.1.8** *Let  $S$  be a partially ordered inverse semigroup. Then  $G = S/\sigma$  is a partially ordered group under the partial order defined by*

$$X \leq Y \text{ if and only if } x \leq y \text{ for some } x \in X, y \in Y.$$

*The canonical map  $\sigma^\natural : S \rightarrow G$  is isotone. If  $S$  is a  $\vee$ -semilattice semigroup [or a  $\wedge$ -semilattice semigroup] then  $G$  is a lattice ordered group and  $\sigma^\natural$  preserves the semilattice operations*

$$(a \vee b)\sigma^\natural = a\sigma^\natural \vee b\sigma^\natural \qquad [(a \wedge b)\sigma^\natural = a\sigma^\natural \wedge b\sigma^\natural]$$

**Proof.** First, we show that the relation  $\leq$  is a partial order on  $G$ .

Suppose that  $X \leq Y, Y \leq Z$ . Then there exist  $x \in X, y_1, y_2 \in Y, z \in Z$  such that  $x \leq y_1, y_2 \leq z$ . Since  $y_1, y_2 \in Y$  there exists  $e = e^2 \in S$  with  $ey_1 = ey_2$  so that  $ex \leq ey_1 = ey_2 \leq ez$  and further, from the definition of  $\sigma$ ,  $ex \in X, ez \in Z$ . Thus  $X \leq Z$ .

Next, if  $X \leq Y$  and  $Y \leq X$ , there exist  $x_1, x_2 \in X$ ,  $y_1, y_2 \in Y$  with  $x_1 \leq y_1$ ,  $y_2 \leq x_2$ . Since  $x_1, x_2 \in X$ ,  $y_1, y_2 \in Y$  there exist  $e^2 = e$ ,  $f^2 = f$  such that  $ey_1 = ey_2$ ,  $fx_1 = fx_2$ . Then  $ex_1 \leq ey_1 = ey_2 \leq ex_2$  which, since idempotents commute, gives

$$efx_1 \leq efy_1 = efy_2 \leq ef x_2 = ef x_1$$

so that  $efx_1 = efy_1$  and so  $X = Y$ .

We now show that  $G$  is a partially ordered group under this partial order. If  $X_1 \leq Y_1, X_2 \leq Y_2$  then there exist  $x_i \in X_i, y_i \in Y_i$  such that  $x_i \leq y_i, i = 1, 2$ . Then  $x_1x_2 \leq y_1y_2$  so that  $X_1X_2 \leq Y_1Y_2$ . Thus  $G = S/\sigma$  is a partially ordered group and also  $\sigma^\natural : S \rightarrow G$  is an isotone (order preserving) homomorphism.

Next suppose that  $S$  is a  $\vee$ -semilatticed semigroup and let  $X, Y \in G$  with  $x \in X, y \in Y$ . Then  $y, x \leq x \vee y$  implies  $X, Y \leq (x \vee y)\sigma^\natural$ . On the other hand  $X, Y \leq Z$  implies  $x_1 \leq z_1, y_1 \leq z_2$  for some  $x_1 \in X, y_1 \in Y, z_1, z_2 \in Z$ . Then  $ez_1 = ez_2$  for some idempotent  $e$  so that  $ex_1 \vee ey_1 \leq ez_1$ . But  $f_1x = f_1x_1, f_2y = f_2y_2$  for some idempotents  $f_1, f_2$ . Thus  $ef_1f_2x = ef_1f_2x_1$  and  $ef_1f_2y = ef_1f_2y_1$  so that

$$ef_1f_2(x \vee y) = ef_1f_2(x_1 \vee y_1) = ef_1f_2x_1 \vee ef_1f_2y_1 \leq ef_1f_2z \in Z.$$

Hence  $(x \vee y)\sigma^\natural \leq Z$  and so  $G$  is a semilattice under  $\leq$  and  $\sigma^\natural$  is a  $\vee$ -semilattice homomorphism as well as a semigroup homomorphism. It further follows that

$$\begin{aligned} a\sigma^\natural(b\sigma^\natural \vee c\sigma^\natural) &= a\sigma^\natural(b \vee c)\sigma^\natural \\ &= (a(b \vee c))\sigma^\natural \\ &= (ab \vee ac)\sigma^\natural \\ &= a\sigma^\natural b\sigma^\natural \vee a\sigma^\natural c\sigma^\natural \end{aligned}$$

that is  $A(B \vee C) = AB \vee AC$  and similarly  $(B \vee C)A = BA \vee CA$ . Finally, since  $\wedge$  can be defined in terms of  $\vee$  in partially ordered groups,  $G$  is a lattice ordered group.  $\square$

## 3.2 Totally Ordered Inverse Semigroups

In this section we describe results, essentially due to T. Saito, on the structure of totally ordered inverse semigroups. As in previous sections, we will primarily restrict our attention to  $E$ -unitary inverse semigroups although

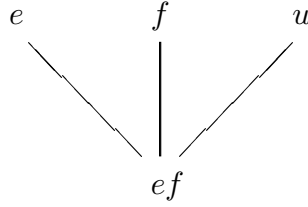
Sâito has also analyzed totally ordered inverse semigroups which are not  $E$ -unitary. It is interesting to note that the concept, though not the name, of  $E$ -unitary inverse semigroups was introduced by Sâito in his study of totally ordered inverse semigroups.

**Proposition 3.2.1** *Let  $S$  be a totally ordered inverse semigroup. Then, under the natural partial order  $\preceq$  the set  $E$  of idempotents of  $S$  is a binary tree*

**Proof.** Let  $e, f, u$  be idempotents of  $S$  and suppose that  $f, u \preceq e$  under the natural partial order; that is,  $f = ef = fe$  and  $u = ue = eu$ . Then, without loss of generality we may suppose that  $f \leq u$  in the imposed ordering. There are two possibilities, since  $S$  is totally ordered: (i)  $fu \leq e$  and (ii)  $e \leq fu$ .

In case (i), we have  $f = f.f \leq fu = f.fu \leq fe = f$ . Hence  $f = fu$  and  $f \preceq u$ . In case (ii),  $u = u.u \geq fu = f.u.u \geq eu = u$ . Hence  $u = fu$  and  $u \preceq f$ . This means that  $E$  is a tree under the natural ordering.

Now suppose that there exists idempotents  $e, f, u$  such that  $ef = fu = ue$ . Then, for example,  $f \geq u \geq e$  implies  $fu \geq u \geq ue$  so that  $u = fu$ . Hence  $E$  cannot contain a partially ordered subset isomorphic to



That is  $E$  is a binary tree.  $\square$

Proposition 3.2.1 actually shows that a semilattice can be totally ordered only if it is a binary tree. The converse is also true. That is, any binary tree can be turned into a totally ordered semilattice. Before turning to the proof of this result we shall characterize  $E$ -unitary inverse semigroups which can be totally ordered.

**Lemma 3.2.2** *Let  $S$  be an  $E$ -unitary inverse semigroup. Then  $(x, y) \in \sigma$  if and only if  $xx^{-1}y = yy^{-1}x$ .*

**Proof.** Suppose  $xx^{-1}y = yy^{-1}x$ . Then  $ey = ex$  for  $e^2 = e = xx^{-1}yy^{-1}$  so that  $(x, y) \in \sigma$ . Conversely, suppose  $(x, y) \in \sigma$ . Then  $(x^{-1}, y^{-1}) \in \sigma$  and so  $ex^{-1} = ey^{-1}$  for some idempotent  $e$ . This implies  $ex^{-1}y = ey^{-1}y$  is idempotent and so, because  $S$  is  $E$ -unitary,  $x^{-1}y$  is idempotent. It follows that  $x^{-1}y = (x^{-1}y)(x^{-1}y)^{-1} = x^{-1}yy^{-1}x$  and so

$$xx^{-1}y = x.x^{-1}yy^{-1}x = yy^{-1}.xx^{-1}x = yy^{-1}x$$

because idempotents commute in  $S$ .  $\square$

**Corollary 3.2.3** *Let  $S$  be an inverse semigroup. Then  $S$  is  $E$ -unitary if and only if  $\mathcal{R} \cap \sigma = \Delta$  the equality relation.*

**Proof.** Suppose that  $S$  is  $E$ -unitary and that  $(x, y) \in \mathcal{R} \cap \sigma$ . Then  $xx^{-1} = yy^{-1}$  and so by the Lemma,  $x = y$ . Conversely, suppose that  $ex = e$  for some idempotent  $e$ . Then  $(x, xx^{-1}) \in \mathcal{R} \cap \sigma$  so  $x = xx^{-1}$  is idempotent.  $\square$

**Proposition 3.2.4** *Let  $S$  be a totally ordered  $E$ -unitary inverse semigroup. Then, for  $x, y \in S$ ,*

$$x \leq y \text{ if and only if } x\sigma^{\natural} < y\sigma^{\natural} \text{ or } (x, y) \in \sigma \text{ and } xx^{-1} \leq yy^{-1}.$$

**Proof.** Define a partial order  $\leq$  on  $E \times G$  by  $(e, g) \leq (f, h)$  if and only if  $g < h$  or  $g = h$  and  $e \leq f$ . Then, since  $G$  and  $E$  are totally ordered, it is easy to see that  $\leq$  is a total order. Further, from Corollary 3.2.3, the map  $x \mapsto (xx^{-1}, x\sigma^{\natural})$  is a bijection of  $S$  onto  $\{(xx^{-1}, x\sigma^{\natural}) : x \in S\}$ . Denote the inverse of this bijection by  $\phi$ . We show that  $\phi$  is order preserving. Then, since each of  $S$  and  $\{(xx^{-1}, x\sigma^{\natural}) : x \in S\}$  is totally ordered, this makes  $\phi$  an order isomorphism and so, in  $S$ ,

$$x \leq y \text{ if and only if } x\sigma^{\natural} < y\sigma^{\natural} \text{ or } x\sigma^{\natural} = y\sigma^{\natural} \text{ and } xx^{-1} \leq yy^{-1}.$$

Suppose  $x\sigma^{\natural} < y\sigma^{\natural}$  then, since  $\sigma^{\natural}$  is order preserving,  $y \not\leq x$ . Hence, since  $S$  is totally ordered,  $x < y$ . Now suppose that  $x\sigma^{\natural} = y\sigma^{\natural}$  and  $xx^{-1} \leq yy^{-1}$ . Then, because  $S$  is  $E$ -unitary,  $xx^{-1}y = yy^{-1}x$  which implies

$$x = xx^{-1}x \leq yy^{-1}x = xx^{-1}y \leq yy^{-1}y = y.$$

Hence  $\phi$  is order preserving as required.  $\square$

**Corollary 3.2.5** *Let  $S$  be a totally ordered  $E$ -unitary inverse semigroup. If  $(x, y) \in \sigma$  then  $xx^{-1} \leq yy^{-1}$  if and only if  $x^{-1}x \leq y^{-1}y$ .*

Every  $E$ -unitary inverse semigroup  $S$  is isomorphic to a  $P$ -semigroup  $P(G, \mathcal{X}, \mathcal{Y})$  where  $G \approx S/\sigma$  and  $\mathcal{Y}$  is isomorphic to the semilattice  $E$  of idempotents of  $S$ . If  $S$  is totally ordered then  $G$  is a totally ordered group and  $\mathcal{Y}$  is a totally ordered semilattice — thus a binary tree. Since  $\mathcal{X} = G\mathcal{Y}$  and  $\mathcal{X}$  is down directed, it follows that  $\mathcal{X}$  is a tree.

**Theorem 3.2.6** *Let  $G$  be a totally ordered group acting on a tree  $\mathcal{X}$  and let  $\mathcal{Y}$  be an ideal of  $\mathcal{X}$  which is a binary tree and which is such that  $\mathcal{X} = G\mathcal{Y}$ . Suppose further that  $\mathcal{Y}$  is a totally ordered semilattice. Then  $P(G, \mathcal{X}, \mathcal{Y})$  is a totally ordered  $E$ -unitary inverse semigroup under*

$$(a, g) \leq (b, h) \text{ if and only if } g < h \text{ or } g = h \text{ and } a \leq b$$

*if and only if, for  $a, b \in \mathcal{Y}$ ,  $g \in G$  such that  $ga, gb \in \mathcal{Y}$ ,  $a \leq b$  if and only if  $ga \leq gb$ .*

**Proof.** Suppose that  $S = P(G, \mathcal{X}, \mathcal{Y})$  is totally ordered under  $\leq$ . Then, for  $a, b \in \mathcal{Y}$ ,  $g \in G$  such that  $ga, gb \in \mathcal{Y}$ ,  $x = (a, g^{-1})$ ,  $y = (b, g^{-1}) \in S$ . Since  $(x, y) \in \sigma$ ,  $xx^{-1} \leq yy^{-1}$  if and only if  $x^{-1}x \leq y^{-1}y$ . That is,  $a \leq b$  if and only if  $ga \leq gb$  so the compatibility condition in the statement of the theorem holds.

Conversely, suppose that this compatibility condition holds. Then, since  $\leq$  is clearly a total order on the set  $S$  it is only necessary to prove that  $\leq$  is compatible with multiplication. Indeed, since  $g < h$  implies  $kg < kh$  and  $gk < hk$ , we need only show that  $a \leq b$  and  $(a, g), (b, g) \in S$  implies  $(a, g)(c, k) \leq (b, g)(c, k)$  and  $(c, k)(a, g) \leq (c, k)(b, g)$  for each  $(c, k) \in S$ . That is, we need only show that  $a \wedge gc \leq b \wedge gc$  and  $c \wedge ka \leq c \wedge kb$ .

But  $(a, g), (b, g) \in S$  and  $a \leq b$  implies  $g^{-1}a, g^{-1}b \in \mathcal{Y}$  and so  $g^{-1}a \leq g^{-1}b$ . Whence, since  $E$  is a semilatticed semigroup under  $\leq$ ,  $g^{-1}a \wedge c \leq g^{-1}b \wedge c$  with both  $g^{-1}a \wedge c$  and  $g^{-1}b \wedge c \in \mathcal{Y}$ . Thus since  $a \wedge gc, b \wedge gc \in \mathcal{Y}$ ,  $g^{-1}a \wedge c \leq g^{-1}b \wedge c$  implies  $a \wedge gc = g(g^{-1}a \wedge c) \leq g(g^{-1}b \wedge c) = b \wedge gc$ .

Likewise,  $(c, k) \in S$  implies  $k^{-1}c \in \mathcal{Y}$  and so, since  $a \leq b$ ,  $k^{-1}c \wedge a \leq k^{-1}c \wedge b$ . But then, since  $c \wedge ka, c \wedge kb \in \mathcal{Y}$ , we also have

$$c \wedge ka = k(k^{-1}c \wedge a) \leq k(k^{-1}c \wedge b) = c \wedge kb.$$

Hence  $S$  is a totally ordered inverse semigroup under  $\leq$  as required.  $\square$

The next proposition gives a method for turning a binary tree into a totally ordered semilattice. We indicate just one method. Sâito discusses the general situation. Furthermore we give an informal argument, based on the graph of the tree, rather than the full details of the proof.

**Proposition 3.2.7** *Let  $E$  be a semilattice then  $E$  can be totally ordered if and only if  $E$  is a binary tree.*

**Proof.** We have already seen that a totally ordered semilattice is a binary tree so suppose that  $E$  is a binary tree.

The first step is to assign an order to the branches of the tree. Recall that an element  $e$  is a branch point if there exist  $u, v \in E$  with  $e, u, v$  distinct and  $e = uv$ . Then  $u$  belongs to one branch at  $e$  and  $v$  belongs to the other. Arbitrarily assign the branch number 0 to one branch at  $e$  and 1 to the other - or *left* to one branch and *right* to the other.

Now define  $\leq$  on  $E$  as follows:

$$e \leq f \text{ if and only if } \begin{cases} \text{if } e \preceq f \text{ if neither } e \text{ nor } f \text{ is a branch point} \\ \text{if } e \text{ is a branch point and } f \text{ is in the right branch at } e \\ \text{if } f \text{ is a branch point and } e \text{ is in the left branch at } f \\ e \text{ if } ef \notin \{e, f\} \text{ and } e \text{ is in the left branch at } ef. \end{cases}$$

Then, by looking at the graph of  $E$ , one can see that  $\leq$  is a total order on  $E$ .

Suppose that  $e \leq f$  and let  $g \in E$ . If  $e \preceq f$  then  $eg \preceq fg$  for every  $g \in E$  so that to show compatibility of multiplication with  $\leq$  we need only consider the other three cases.

In each of these cases, if  $g \preceq ef$ , then  $eg = fg$  so that  $eg \leq fg$ . Hence we may assume  $g \not\preceq ef$ . Consider the second case;  $e$  is a branch point. Then  $g$  is in either the left or the right branch at  $e$ . If  $g$  is in the left branch at  $e$  then  $eg = e = fg$  so that  $eg \leq fg$ . While if  $g$  is in the right branch,  $eg = e$  and  $fg$  is also in the right branch at  $e$ . Thus again  $eg \leq fg$ . The next case where  $f$  is a branch point, is similar.

Finally consider the case where  $ef \notin \{e, f\}$ . Since  $g \not\preceq ef$  either  $g$  is in the left branch at  $ef$  or  $g$  is in the right branch at  $ef$ . In the first case,  $eg$  is in the left branch while  $fg = ef$  so that from the definition of  $\leq$ ,  $eg \leq fg$ . In the second case,  $eg = ef$  while  $fg$  is in the right branch at  $ef$  so again  $eg \leq fg$ . Hence multiplication is compatible with  $\leq$ .  $\square$

It is an easy matter to see that the idempotents of the free monogenic inverse semigroup do not form a tree. For example,  $a^2a^{-2} \preceq aa^{-1}$  and  $aa^{-1}a^{-1}a \preceq aa^{-1}$  in the free inverse semigroup on  $\{a\}$  but neither of  $a^2a^{-2}$  nor  $aa^{-1}a^{-1}a$  is comparable in the natural partial order. It follows that the same is true for the free inverse semigroup on any non-empty set. Hence no free inverse semigroup can be totally ordered. This differs from the situation with free groups. Every free group can be totally ordered but no free group admits a lattice ordering which is not a total ordering. No free inverse semigroup can be totally ordered but the free monogenic inverse semigroup admits a lattice ordering.

**Theorem 3.2.8** *Let  $G$  be a totally ordered group, let  $\mathcal{X}$  be the set of all intervals  $[a, b]$  with  $a \leq b \in G$  and let  $\mathcal{Y} = \{[a, b] : a \leq 1 \leq b\}$ . Then  $\mathcal{X}$  is a down directed partially ordered set under containment with  $\mathcal{Y}$  as an ideal and subsemilattice. Further,  $G$  acts on  $\mathcal{X}$  by left multiplication in such a way that  $\mathcal{X} = G\mathcal{Y}$ . The  $E$ -unitary inverse semigroup  $P(G, \mathcal{X}, \mathcal{Y})$  is a lattice ordered semigroup under*

$$([a, b], g) \leq ([c, d], h) \text{ if and only if } g < h \text{ or } g = h \text{ and } [a, b] \supseteq [c, d].$$

**Proof.** For  $([a, b], g), ([c, d], g) \in P$ ,

$$\begin{aligned} ([a, b], g) \vee ([c, d], g) &= ([a \vee c, b \wedge d], g) \\ ([a, b], g) \wedge ([c, d], g) &= ([a \wedge c, b \vee d], g). \end{aligned}$$

Since elements of  $P$  with distinct second coordinates are comparable, to prove that  $P$  is a lattice ordered semigroup we need only prove that the equations above are compatible with multiplication.

Let  $([u, v], k) \in P$ . Then

$$([u, v], k)([a, b], g) = ([u, v] \vee k[a, b], kg) = ([u \wedge ka, v \vee kb], kg).$$

Thus  $\vee$  is compatible with left multiplication if and only if

$$[u \wedge ka, v \vee kb] \vee [u \wedge kc, v \vee kd] = [u \wedge k(a \vee c), v \vee k(b \wedge d)].$$

But, since  $G$  is a chain (thus a distributive lattice) and multiplication is an order automorphism,

$$(u \wedge ka) \vee (u \wedge kc) = u \wedge (ka \vee kc) = u \wedge k(a \vee c)$$

and  $(v \vee kb) \wedge (v \vee kd) = v \vee k(b \wedge d)$ .

Likewise  $\wedge$  is compatible with left multiplication if and only if

$$[u \wedge ka, v \vee kb] \wedge [u \wedge kc, v \vee kd] = [u \wedge k(a \wedge c), v \vee k(b \vee d)].$$

Since  $(u \wedge ka) \wedge (u \wedge kc) = u \wedge k(a \wedge c)$  and  $(v \vee kb) \vee (v \vee kd) = v \vee k(b \vee d)$  the equation above holds. Thus compatibility of left multiplication holds.

As for right multiplication, we need to check that

$$\begin{aligned} ([a, b] \vee g[u, v]) \vee ([c, d] \vee g[u, v]) &= [a \vee c, b \wedge d] \vee g[u, v] \\ ([a, b] \vee g[u, v]) \wedge ([c, d] \vee g[u, v]) &= [a \wedge c, b \vee d] \vee g[u, v]. \end{aligned}$$

The first of these is

$$[a \vee gu, b \wedge gv] \vee [c \vee gu, d \wedge gv] = [(a \vee c) \vee gu, (b \wedge d) \wedge gv].$$

That is

$$[(a \vee gu) \vee (c \vee gu), (b \wedge gv) \wedge (d \wedge gv)] = [(a \vee c) \vee gu, (b \wedge d) \wedge gv].$$

Since  $G$  is a lattice, this is immediate. As for the second, it is equivalent to

$$[a \vee gu, b \wedge gv] \wedge [c \vee gu, d \wedge gv] = [(a \wedge c) \vee gu, (b \vee d) \wedge gv]$$

which, since  $G$  is a distributive lattice, is also true. Hence  $P$  is a lattice ordered semigroup.  $\square$

**Corollary 3.2.9** *Let  $S = G^+$  be the positive cone of a non-trivial totally ordered group. Then  $E(S) = I(S)$  can be lattice ordered but it cannot be totally ordered.*

**Proof.** In the terminology of the Theorem,  $E(S) \approx P(G, \mathcal{X}, \mathcal{Y})$  so it can be lattice ordered. On the other hand, its idempotents are a distributive lattice but not a chain. Hence it cannot be totally ordered.  $\square$

**Corollary 3.2.10** *Let  $S$  be the free inverse monoid on the generator  $a$  then  $S$  is a lattice ordered inverse semigroup under the partial ordering given by*

$$a^r a^{-s} a^t \leq a^u a^{-v} a^w \text{ if and only if } \begin{cases} \text{either} & r + t - s < u + w - v \\ \text{or} & r + t - s = u + w - v, t \geq w, r \geq u. \end{cases}$$

### 3.3 Amenably Ordered Inverse Semigroups

When we were studying bisimple inverse monoids and inverse monoids separated over a subsemigroup we saw that the following relations played a natural role

$$\begin{aligned} (i) \quad & a^{-1}a \wedge b^{-1}b = (a \wedge b)^{-1}(a \wedge b) \\ (ii) \quad & aa^{-1} \wedge bb^{-1} = (a \wedge b)(a \wedge b)^{-1} \end{aligned}$$

These conditions reflect the relation between the natural partial order on the idempotents of the inverse semigroup and the partial order of the associated lattice ordered group. It is natural to consider these conditions too in the context of lattice ordered inverse semigroups.

**Definition 3.3.1** *A partially ordered inverse semigroup  $S$  is left amenably ordered if  $a \leq b$  implies  $a^{-1}a \leq b^{-1}b$ . It is naturally ordered if  $e \preceq f$  implies  $e \leq f$ .*

**Lemma 3.3.2** *Let  $S$  be a naturally ordered inverse semigroup. Then the partial order on the idempotents coincides with the natural partial order.*

**Proof.** Suppose  $e \leq f$ . Then  $e \leq ef \preceq e$ . Hence, since  $S$  is naturally ordered,  $e \leq ef \leq e$  and so  $e = ef$ . That is  $e \preceq f$ .  $\square$

It is usually somewhat difficult to check whether or not a partially ordered inverse semigroup is a lattice ordered semigroup. Of course, it must be a lattice if this is to happen. But in general this is not enough to guarantee that a partially ordered inverse semigroup is a lattice ordered semigroup although it is enough for groups. Indeed a  $\vee$ -semilatticed semigroup can be a lattice without being a lattice ordered semigroup. However

**Theorem 3.3.3** *Let  $S$  be a  $\vee$ -semilatticed inverse semigroup. Then  $S$  is left amenably and naturally ordered if and only if, for each  $a, b \in S$ ,  $a \in S(a \vee b)$ . In this case,  $S$  is a lattice ordered inverse semigroup and for each  $a, b \in S$ ,*

$$\begin{aligned}(a \vee b)^{-1}(a \vee b) &= a^{-1}a \vee b^{-1}b \\ (a \wedge b)^{-1}(a \wedge b) &= a^{-1}a \wedge b^{-1}b.\end{aligned}$$

*Further*

$$a \wedge b = (a^{-1} \vee b^{-1})a^{-1}ab^{-1}b = a(a \vee b)^{-1}b \vee b(a \vee b)^{-1}a.$$

**Corollary 3.3.4** *Let  $S$  be an  $E$ -unitary  $\vee$ -semilatticed inverse semigroup. If  $S$  is left amenably and naturally ordered then  $S$  is a distributive lattice.*

**Proof.** Suppose that  $a \vee b = a \vee c$  and  $a \wedge b = a \wedge c$ . Then  $(a \vee b)\sigma^{\natural} = (a \vee c)\sigma^{\natural}$  and  $(a \wedge b)\sigma^{\natural} = (a \wedge c)\sigma^{\natural}$ . Whence, since  $\sigma$  is a lattice homomorphism and  $G = S/\sigma$  is a lattice ordered group,  $(b, c) \in \sigma$ .

Next  $a \vee b = a \vee c$  and  $a \wedge b = a \wedge c$  imply

$$\begin{aligned}a^{-1}a \vee b^{-1}b &= (a \vee b)^{-1}(a \vee b) = (a \vee c)^{-1}(a \vee c) = a^{-1}a \vee c^{-1}c \\ a^{-1}a \wedge b^{-1}b &= (a \wedge b)^{-1}(a \wedge b) = (a \wedge c)^{-1}(a \wedge c) = a^{-1}a \wedge c^{-1}c.\end{aligned}$$

Hence, since the idempotents form a distributive lattice,  $b^{-1}b = c^{-1}c$ ; that is  $b\mathcal{L}c$ . But then  $(b, c) \in \sigma \cap \mathcal{L}$  and so, because  $S$  is  $E$ -unitary,  $b = c$ . Thus  $S$  is a distributive lattice.  $\square$

We have seen that the minimum group congruence  $\sigma$  on an inverse semigroup has an important role to play in the study of totally ordered inverse semigroups as well as in the study of inverse semigroups in general. There is an other important congruence on an arbitrary inverse semigroup. This is the maximum idempotent separating congruence  $\mu$  which is defined by

$$(a, b) \in \mu \text{ if and only if } a^{-1}ea = b^{-1}eb \text{ for each idempotent } e.$$

Equivalently,  $(a, b) \in \mu$  if and only if  $ea\mathcal{L}eb$  for each idempotent  $e$ . This congruence also arises naturally when one considers left amenable semilattice ordered inverse semigroups.

**Lemma 3.3.5** *Let  $S$  be a  $\wedge$ -semilatticed inverse semigroup in which  $a^{-1}a \wedge b^{-1}b = (a \wedge b)^{-1}(a \wedge b)$  for each  $a, b \in S$ . Then  $\mu$  is a  $\wedge$ -semilattice congruence on  $S$ , as well as a semigroup congruence.*

**Proof.** We show that  $(a, b) \in \mu$  implies  $(a \wedge c, b \wedge c) \in \mu$  for each  $c \in S$ . Let  $e \in E$ . Then

$$\begin{aligned} (a \wedge c)^{-1}e(a \wedge c) &= [e(a \wedge c)]^{-1}[e(a \wedge c)] \\ &= (ea \wedge ec)^{-1}(ea \wedge ec) \\ &= (ea)^{-1}(ea) \wedge (ec)^{-1}(ec) \\ &= (eb)^{-1}(eb) \wedge (ec)^{-1}(ec) \text{ since } (a, b) \in \mu \\ &= (b \wedge c)^{-1}e(b \wedge c). \end{aligned}$$

Thus  $(a \wedge c)\mu(b \wedge c)$  and so  $\mu$  is a  $\wedge$ -semilattice congruence.  $\square$

**Proposition 3.3.6** *Let  $S$  be a  $\wedge$ -semilatticed inverse semigroup in which  $a^{-1}a \wedge b^{-1}b = (a \wedge b)^{-1}(a \wedge b)$  for each  $a, b \in S$ . Then  $S/\mu$  is  $\wedge$ -semilatticed inverse semigroup in which  $A^{-1}A \wedge B^{-1}B = (A \wedge B)^{-1}(A \wedge B)$  for each  $A, B \in S/\mu$ ;*

$$A \leq B \text{ if and only if } a^{-1}ea \leq b^{-1}eb$$

for some [all]  $a \in A, b \in B$  and all  $e \in E$ .

**Proof.** From Lemma 3.3.5,  $S/\mu$  is a  $\wedge$ -semilattice. Let  $A, B, C \in S/\mu$  with  $a \in A, b \in B, c \in C$ . Then

$$A(B \wedge C) = a\mu^\natural(b \wedge c)\mu^\natural = [a(b \wedge c)]\mu^\natural = (ab \wedge ac)\mu^\natural = ab\mu^\natural \wedge ac\mu^\natural = AB \wedge AC$$

since  $\mu$  is both a semigroup and semilattice congruence. Hence  $S/\mu$  is a  $\wedge$ -semilatticed semigroup and clearly obeys the amenability identity.

Finally,  $A \leq B$  if and only if  $A = A \wedge B$ . That is if and only if, for some [all]  $a \in A, b \in B$ ,  $(a, a \wedge b) \in \mu$ . Thus if and only if

$$a^{-1}ea = (a \wedge b)^{-1}e(a \wedge b) = (ea \wedge eb)^{-1}(ea \wedge eb) = a^{-1}ea \wedge b^{-1}eb$$

equivalently  $a^{-1}ea \leq b^{-1}eb$  for all  $e \in E$ .  $\square$

In case  $S$  is not only left amenable but also right amenable much stronger results hold.

**Lemma 3.3.7** *Let  $S$  be a left amenable partially ordered inverse semigroup and suppose that  $a \leq b$  where  $a\mathcal{L}b$ . Then  $bb^{-1} \leq aa^{-1}$ .*

**Proof.** Since  $a \leq b$  it follows that  $b^{-1}a \leq b^{-1}b = a^{-1}a$  and thus  $(b^{-1}a)^{-1}(b^{-1}a) \leq a^{-1}a$ . That is  $a^{-1}bb^{-1}a \leq a^{-1}a$  from which  $aa^{-1}bb^{-1} \leq aa^{-1}$  by the compatibility of multiplication. But also  $a \leq b$  implies  $a^{-1}a \leq a^{-1}b$  which gives  $b^{-1}b = a^{-1}a \leq b^{-1}aa^{-1}b$  so that  $bb^{-1} \leq bb^{-1}aa^{-1} \leq aa^{-1}$ .  $\square$

**Theorem 3.3.8** *Let  $S$  be a  $\vee$ -semilatticed inverse semigroup in which  $a^{-1}a \vee b^{-1}b = (a \vee b)^{-1}(a \vee b)$  for each  $a, b \in S$ . If  $S$  is further right amenably ordered then  $S$  is a semilattice of groups.*

**Proof.** Suppose  $a \mathcal{L} b$  then  $a^{-1}a = b^{-1}b$  so that  $a \mathcal{L}(a \vee b)$ . Hence, by the Lemma,  $(a \vee b)(a \vee b)^{-1} \leq aa^{-1}$ . But  $a \leq b$  implies  $aa^{-1} \leq (a \vee b)(a \vee b)^{-1}$  since  $S$  is right amenable. Thus  $a \mathcal{R}(a \vee b)$  and similarly  $b \mathcal{R}(a \vee b)$ . Hence  $a \mathcal{H} b$ . Since  $S$  is inverse, it follows that  $S$  is a semilattice of groups.  $\square$

**Corollary 3.3.9** *Let  $S$  be a totally ordered inverse semigroup. If  $S$  is amenably ordered then  $S$  is a semilattice of groups.*

If  $S$  is naturally and amenably ordered and is  $\vee$ -semilatticed then  $S$  is a distributive lattice of lattice ordered groups. Its structure can be described in terms of homomorphisms between the lattice ordered groups. It is similar to the structure of semilattices of groups which was described by Clifford. However, in this case two families of homomorphisms are needed — one to describe the multiplication, the other to determine the order and to ensure that it is a lattice ordering. The latter step requires a delicate analysis of the structure of the lattice ordered groups which are involved and, in particular, it uses the fact that the convex  $l$ -subgroups of a lattice ordered group form a distributive lattice; Corollary 1.3.4 and Theorem 1.3.6.

**Theorem 3.3.10** *Let  $E$  be a distributive lattice and, for each  $e \in E$ , let  $G_e$  be a lattice ordered group, with identity  $e$ . Let  $\phi_{e,f} : G_e \rightarrow G_f$  and  $\psi_{f,e} : G_f \rightarrow G_e$  with  $e \geq f$  be directed systems of lattice group homomorphisms such that*

- (i)  $\psi_{f,e} \phi_{e,f} = 1_f$  the identity on  $G_f$ , if  $f \leq e$ ;
- (ii)  $\ker \phi_{e \vee f, e} \cap \ker \phi_{e \vee f, f} = \{e \vee f\}$  for each  $e, f \in E$ ;
- (iii) for  $e, f \in E$  the diagram

$$\begin{array}{ccc}
 & G_{e, e \vee f} & \\
 \psi_{e, e \vee f} \nearrow & & \searrow \phi_{e \vee f, f} \\
 G_e & & G_f \\
 \phi_{e, e \wedge f} \searrow & & \nearrow \psi_{e \wedge f, f} \\
 & G_{e \wedge f} &
 \end{array}$$

*commutes.*

Then  $S = \bigcup\{G_e : e \in E\}$  is an amenably and naturally lattice ordered inverse semigroup under the product

$$a * b = a\phi_{e,e\wedge f}b\phi_{f,e\wedge f} \quad \text{for } a \in G_e, b \in G_f$$

and with

$$a \leq b \text{ if and only if } e \leq f \text{ and } a \leq b\phi_{f,e} \text{ where } a \in G_e, b \in G_f.$$

Conversely, each amenably and naturally latticed ordered inverse semigroup is isomorphic to one of this form.

The conditions in the statement of this theorem are quite complicated. They simplify considerably in the case when  $S$  is  $E$ -unitary. For then the linking homomorphisms  $\phi_{e,f}$  are one-to-one and, by (i) onto. Hence they are lattice group isomorphisms and the family  $\psi_{f,e}$  is the family of inverse isomorphisms. In this case, we get

**Proposition 3.3.11** *Let  $S$  be a partially ordered inverse semigroup.. Then the following conditions are equivalent:*

- (i)  $S$  is isomorphic to the direct product of a distributive lattice and a lattice ordered group;
- (ii)  $S$  is a naturally and amenably lattice ordered  $E$ -unitary inverse semigroup;
- (iii)  $S$  is a naturally and amenably ordered lattice ordered inverse semigroup in which the identity  $[aa^{-1} \vee (b \wedge cc^{-1})]^2 = aa^{-1} \vee (b \wedge cc^{-1})$  holds.

Finally, suppose that  $S$  is a naturally ordered left amenable  $\vee$ -semilatticed inverse semigroup. Then we saw earlier, Proposition 3.3.6, that the maximum idempotent separating congruence  $\mu$  turns  $S/\mu$  into a fundamental naturally ordered left amenable  $\vee$ -semilatticed inverse semigroup. The kernel of this congruence is the centralizer  $C(E) = \{a \in S : ae = ea \text{ for all idempotents } e \in S\}$ . It is easy to see that  $C(E)$  is sublattice as well as an inverse subsemigroup of  $S$ . It inherits the property of being naturally and left amenably ordered from  $S$ . Since  $C(E)$  is a semilattice of groups it is thus amenably ordered so that its structure is given by Theorem 3.3.10. On the other hand,  $S/\mu$  is isomorphic to a lattice ordered inverse subsemigroup of the semigroup  $T_E$  of isomorphisms between principal ideals of the semilattice  $E$  where for  $\theta, \phi \in T_E$ ,

$$\theta \leq \phi \text{ if and only if } (ae_\theta)\theta \leq (ae_\phi)\phi \text{ for each } a \in E$$

where, for example,  $e_\theta$  denotes the greatest element of the domain of  $\theta$ .

### 3.4 Compatible Orders on the Bicyclic Semigroup

In this short section, we describe the semilattice orderings on the bicyclic semigroup; no proofs are given. By Theorem 3.3.8, we cannot expect these to be amenable orderings.

**Proposition 3.4.1** *Let  $\leq$  be a semilattice order on the bicyclic semigroup  $B = \langle a, b : ab = 1 \rangle$ . Then one of the following four mutually exclusive relations holds in  $B$ :*

- (i)  $a > 1$  and  $ba > 1$ ;
- (ii)  $a < 1$  and  $ba > 1$ ;
- (iii)  $a > 1$  and  $ba < 1$ ;
- (iv)  $a < 1$  and  $ba < 1$ .

**Proof.** Suppose that  $B$  is  $\vee$ -semilattice and let  $x = a \vee 1$ . Then  $x$  commutes with  $a$  and so, from the form of multiplication in  $B$ ,  $x = a^n$  for some  $n$ . Likewise,  $b \vee 1 = b^m$  for some  $m \geq 0$ . But then

$$ab^m = a(b \vee 1) = ab \vee a = 1 \vee a = a^n$$

which implies  $m = 0, n = 1$  or  $m = 1, n = 0$ . Hence either  $a \vee 1 = a$  or  $a \vee 1 = 1$ . Thus  $a > 1$  or  $a < 1$ .

Now consider  $ba \vee 1$ . Then  $a(ba \vee 1) = aba \vee a = a$ . Hence, by Proposition 3.1.7,  $ba \vee 1 = e$  is an idempotent with  $ae = a$ . This implies  $e = 1$  or  $e = ba$ . Thus either  $ba < 1$  or  $ba > 1$ .  $\square$

The bicyclic semigroup is  $E$ -unitary so the total orderings on  $B$  can be described using the results in Section 3.2. There are four of them:

**Theorem 3.4.2** *Let  $B = \langle a, b : ba = 1 \rangle$  be the bicyclic semigroup. Then  $B$  admits exactly four total orderings:*

- (i)  $b^r a^s \leq b^u a^v$  if and only if  $s - r < v - u$  or  $s - r = v - u$  and  $s \leq v$ ;
- (ii)  $b^r a^s \leq b^u a^v$  if and only if  $s - r > v - u$  or  $s - r = v - u$  and  $s \geq v$ ;
- (iii)  $b^r a^s \leq b^u a^v$  if and only if  $s - r > v - u$  or  $s - r = v - u$  and  $s \leq v$ ;
- (iv)  $b^r a^s \leq b^u a^v$  if and only if  $s - r < v - u$  or  $s - r = v - u$  and  $s \geq v$ .

It can be shown that these four total orderings are the only lattice orderings on  $B$ . So, in this sense, the bicyclic semigroup is more like a free group than is the free inverse monoid on one generator. However, there are many more semilattice orderings.

**Theorem 3.4.3** *Let  $B = \langle a, b : ab = 1 \rangle$  be the bicyclic semigroup. Then there are four distinct infinite families of semilattice orderings on  $B$ . Each family contains exactly one total ordering; these are the only lattice orderings on  $B$ .*

Define  $\leq$  by

$$b^r a^s \leq b^y a^x \text{ if and only if } s - r < x - y \text{ or } s - r = y = x \text{ and } s \leq x;$$

then  $\leq$  is one of the four total orderings.

For each  $n \geq 0$ , let  $\leq_n$  be defined by

$$b^r a^s \leq_n b^y a^x \text{ if and only if } y \leq x + (r - s) \text{ and } (n - 1)y \leq nx + \{(n - 1)r - ns\}.$$

Then  $B$  is a  $\vee$ -semilatticed inverse semigroup in which  $a > 1$  and  $ba > 1$ .

Each semilattice ordering on  $B$  is conjugate to  $\leq$  or to exactly one of the  $\leq_n$ . The ordering  $\leq_1$

$$b^r a^s \leq_1 b^y a^x \text{ if and only if } y \leq x + (r - s) \text{ and } s \leq x$$

is left amenable; it is the only left amenable  $\vee$ -semilattice ordering on  $B$ .

Finally, the orderings in these families with  $ba > 1$  are  $\vee$ -semilattice orderings; those with  $ba < 1$  are  $\wedge$ -semilattice orderings.

# Bibliography

- [1] G. BIRKHOFF, "Lattice Theory," Amer. Math. Soc. Colloquium Publications, Vol. 25, Providence, R.I., 1948.
- [2] A. H. CLIFFORD, *A class of  $d$ -simple semigroups*, American J. Math. **15** (1953), 541-556
- [3] M. R. DARNEL, "Theory of Lattice Ordered Groups", Marcel Dekker Inc., New York, N.Y., 1995
- [4] L. FUCHS, "Partially Ordered Algebraic Systems," Pergamon, Oxford, 1963
- [5] P. J. HIGGINS, "Techniques of Semigroup Theory," Oxford University Press, 1992
- [6] J. M. HOWIE, "Fundamentals of Semigroup Theory," London Mathematical Society Monographs, New Series **12**, Oxford, 1996
- [7] D. B. MCALISTER, *Inverse Semigroups Separated over a Subsemigroup*, Trans. American Math. Soc. **182** (1973), 85-117
- [8] D. B. MCALISTER, *Amenably Ordered Inverse Semigroups*, Journal of Algebra, 65 (1980), 118-146
- [9] D. B. MCALISTER, *Compatible Orders on the Bicyclic Semigroup*, Communications in Algebra (to appear Fall 1999)
- [10] M. PETRICH, "Inverse Semigroups", John Wiley Interscience, New York 1984
- [11] TÔRU SAITÔ, *Proper Ordered Inverse Semigroups*, Pacific J. Math., 15 (1965), 649-666
- [12] TÔRU SAITÔ, *Ordered Inverse Semigroups*, Trans. American Math. Soc. **153** (1971), 99-138

- [13] TÔRU SAITÔ, *No Free Inverse Semigroup is Orderable*, Proc. Japan Acad., 50 (1974), 837-838

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