AN INTRODUCTION TO E*-UNITARY INVERSE SEMIGROUPS - FROM AN OLD FASHIONED PERSPECTIVE.

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Thirty years ago, in 1972, H.E. Scheiblich published the first usable description of the free inverse semigroup on a set. This lead to a flowering of research on inverse semigroups, including the structure of E-unitary inverse semigroups. In a similar fashion, the publication of the book "Inverse semigroups: the theory of partial symmetry" by Mark V. Lawson, and, in particular, the chapters on applications of inverse semigroups to tilings have lead to another flowering of research; this time into the theory of E*-unitary inverse semigroups. This talk will give an introduction to the later theory in the light of the earlier perspective that grew from Scheiblich’s result.

1. Introduction and pre-history

2002 is a very good year in the history of semigroups. It is precisely 50 years after the publication of Wagner’s seminal paper [45] on inverse semigroups, “Generalized Groups”, Doklady Akademii Nauk SSSR 84 (1952), 1119-1122. It is also 30 years after the publication of another seminal paper in the theory of inverse semigroups, Scheiblich’s Semigroup Forum paper on the structure of free inverse semigroups, Semigroup Forum 4, (1972), 351-359; [39], [40]. The period between the publication of these two papers, in particular the latter part of the 1960’s had seen a considerable development of information about the structure and properties of inverse semigroups. There were the papers of Munn [27],[29] on fundamental inverse semigroups; there were the beautiful theorems of Reilly [34], and Kochin [11] and Munn [28], on the structure of regular (inverse) semigroups whose idempotents form an α-chain; the structure of more general, but still special, classes of inverse semigroups was investigated, especially by Warne (see for example [46], [47]) and Schreier type extension theorems were developed, for example by Coudron [4] and D’Alarcão [5] for idempotent separating extensions, and by Saitô [36] for group coextensions of inverse semigroups. In principle, it was possible to say that the structure of inverse semigroups was known. But there was a problem with these structure theorems.
The building blocks, full subsemigroups of isomorphisms between principal ideals of a semilattice, and semilattices of groups, were simple and natural. The interrelations between the building blocks were not. The conditions required to describe how extensions were put together were intricate, to say the least. They involved group actions and factor sets which had to obey complicated and stringent interrelated conditions. It was even worse when a congruence wasn’t involved. Frankly, the structure theory of inverse semigroups had reached the point of diminishing returns. One could reasonably say that each new theorem resulted in a gain of information but a loss of insight. A good - or would it be more accurate to say bad - example of this situation can be found in one of my own papers “0-bisimple inverse semigroups” [19] which appeared in the Proceedings of the London Math Soc in 1974 although the work had been done before I left Ireland in 1970. Perhaps the fact that it was four years later before it appeared in print makes the point, that I alluded to above better, than I can.

It was clear that some other approach was needed. I, personally, had been trying to find new constructive ways of describing inverse semigroups, which would avoid factor sets and their complicated interactions, by using cones in partially ordered structures to build inverse semigroups. The idea had been to try to generalize the Clifford [3] and Reilly [35] theory of bisimple inverse semigroups. An example of this was given in the paper [18] which generalized Eberhardt and Sheldon’s insightful characterization of one parameter inverse semigroups [6]. It is interesting that generalizations of both of these approaches have found their way into the modern theory of inverse semigroups, in particular of $E^*$-unitary inverse semigroups, due to Lawson; see for example [14], [15].

But it was Scheiblich’s construction that provided the new direction. The description which he gave of the free inverse semigroup was so implicitly simple, but so powerful, that it lead to a whole new way of looking at inverse semigroups. Earlier structure theorems had taken two, relatively, simple objects a kernel (semilattice of groups) and an image (a fundamental inverse semigroup) and tried to tie them together by means of a Schreier type extension theory. But now that we knew the free inverse semigroup was an essentially - at least conceptually - simple object it became possible to consider a dual approach. On the theory that if an object is simple so are its homomorphic images we could try to construct some conceptually simple inverse semigroups which would have all inverse semigroups as nice (idempotent separating, so that the kernels were nice) homomorphic images. So Scheiblich’s result lead directly to the theory of $E$-unitary inverse
semigroups. The construction of these semigroups reflected Scheiblich’s construction of the free inverse semigroups but the class of semigroups was already known and a Schreir type extension theory had been given for it earlier by T. Saïto [36] who called these semigroups proper. That’s where the $P$ in $E$-unitary inverse semigroups comes from.

2. History: $E$-unitary inverse semigroups.

Let’s recall Scheiblich’s construction [39], [40] of the free inverse monoid $FIM(X)$ on a set $X$. Its elements consist of pairs $(A, g)$ where $A$ is a finite set of reduced words in the free group $FG(X)$ on $X$ which is closed under prefixes, and $g$ is an element of the free group whose reduced form belongs to $A$. Multiplication is defined by

$$(A, g)(B, h) = (A \cup gB, gh).$$

Note that, since $A$ is closed under prefixes, $1 \in A$.

There are a couple of points to be made here. The first relates to the multiplication. This involves translates by elements of the free group of prefix closed subsets. What are these translates? Well to understand this we need to think geometrically. The free group $FG(X)$ is a geometric object or, rather, its Cayley graph is. In this sense, a finite prefix closed subset $B$ is just a convex connected neighborhood of the identity and so, because the group acts by automorphisms on the Cayley graph, $gB$ is just a convex connected neighborhood of any of its members, in particular of $g$. The fact that $g$ belongs to both $A$ and to $gB$ means that $A \cup gB$ is connected and so multiplication is well defined.

The second remark is a philosophical or cultural one. There has always been something unsatisfying about this setup for someone like me, who is a child of the twentieth century and who grew up believing that Einstein’s relativity theory was the most important scientific development of the 1900’s. We can summarize the latter by saying that the Lorentz group acts on space-time and all points which lie on same orbit are equivalent. Well, in Scheiblich’s theorem, we have a group acting but we insist on localizing everything at 1. Relativity would say that we should regard equally all points of the free group. That is we should look at triples $(g, A, h)$ where $A$ is a neighborhood of $g$ and $h$. The triple $(g, A, h)$ with $g, h \in A$ is then equivalent, under the obvious group action to $(1, g^{-1}A, g^{-1}h)$ and so multiplication on equivalence classes of triples can be made to mirror the multiplication of pairs. We shall return to this remark later but notice that an equivalence class of the triples admits an easy interpretation as a
directed graph over \( X \) with a pair of identified vertices. That is, it corresponds to a Munn tree and what we have is an interpretation of Munn’s graphical version \([30], [31]\) of the free inverse semigroup on \( X \).

In any case, the philosophy behind the original study of \( E \)-unitary inverse semigroups was, as I have said, the following: construct a family of inverse semigroups from simple, familiar, naturally related objects in such a way that every inverse semigroup is a nice homomorphic image of a member of the family. Thus there are two theorems. Before stating them, we recall the definition of an \( E \)-unitary inverse semigroup:

**Definition 2.1.** Let \( S \) be an inverse semigroup. Then \( S \) is said to be \( E \)-unitary if and only if \( ab = b \) implies \( a^2 = a \).

This isn’t the usual definition - \( e \leq a \) where \( e \) is idempotent, implies \( a^2 = a \) - but is easily seen to be equivalent to it.

**Theorem 2.1.** Let \( X \) be a down directed partially ordered set with \( Y \) an order ideal and subsemilattice of \( X \). Also let \( G \) be a group which acts on \( X \) by order automorphisms in such a way that \( X = GY \). Then

\[
P = P(G, X, Y) = \{(a, g) \in Y \times G : g^{-1}a \in Y\}
\]

is an \( E \)-unitary inverse semigroup under the multiplication

\[
(a, g)(b, h) = (a \land gb, gh).
\]

Conversely, if \( S \) is an \( E \)-unitary inverse semigroup then \( S \cong P(G, X, Y) \) for a unique (up to isomorphism and equivalence of group actions) \( G, X, \) and \( Y \).

Note that we have here both an existence and a uniqueness theorem for \( E \)-unitary inverse semigroups. All three components \( G, X, \) and \( Y \) are intrinsic to the semigroup \( S \). The group \( G \) is (isomorphic to) the maximum group homomorphic image of \( S \), the semilattice \( Y \) is (isomorphic to) the semilattice of idempotents of \( S \). The role of \( X \) is not so clear but it is none the less intrinsic to \( S \).

**Theorem 2.2.** Let \( S \) be an inverse semigroup. Then \( S \) is an idempotent separating homomorphic image of an \( E \)-unitary inverse semigroup.

I was very proud of these theorems, and particularly the first one, when I obtained them. I wrote to Al Clifford telling him about it and asking him if he would like to see a copy of the manuscript before I submitted it for publication. As I remember, it was some time before he responded
but when he did, the answer was short. "That can't possibly be true", he wrote. Well that was a bit of a blow to the ego after a lot of hard work over quite a period of time!

But I can understand his reaction. The structure of these inverse semigroups appeared to be much more simple than one would have expected on the basis of other structure theorems for special classes of inverse semigroups; for example the structure theorem for 0-bisimple inverse semigroups [19] which I mentioned earlier. I was fortunate I hadn't sent Al some of the early drafts of the proof. Then he would never have believed the result. But I am pretty sure that it is true.

There have been several different proofs since that first one. My personal favorite is the proof due to Douglas Munn [32]. The main part of it consists in identifying the partially ordered set $\mathcal{X}$ since the group $G$ and the semilattice $\mathcal{Y}$ have natural interpretations. In the original proof of the theorem, $\mathcal{X}$ was constructed by a painstaking analysis of how the maximum group homomorphic image acts on $\mathcal{R}$-classes.

Munn's approach was much more external than my internal approach. Let us review it. From the requirement that $\mathcal{X} = G\mathcal{Y}$ we know that each element of $\mathcal{X}$ has the form $ga$ for some $g \in G$ and $a \in \mathcal{Y}$. Of course the expression of the elements $ga$ may not be unique and the problem is to identify when $ga = hb$. More precisely, we want to know when $ga \leq hb$ in order to be able to describe the partial order on $\mathcal{X}$. Well, let's see, suppose that $S = P(G, \mathcal{X}, \mathcal{Y})$. We have

$$ga \leq hb \quad \text{if and only if} \quad h^{-1}ga \leq b$$

$$\quad \text{if and only if} \quad s = (a, g^{-1}h) \in P \quad \text{and} \quad s^{-1}s \leq b.$$ 

Thus, for an $E$-unitary inverse semigroup $S$, we define a quasi-order $\preceq$ on $G \times E$ by

$$(g, a) \preceq (h, b) \iff \exists s \in S \text{ with } ss^{-1} = a, s^{-1}s \leq b \text{ and } s\sigma^2 = g^{-1}h,$$

where $\sigma^2$ denotes the canonical homomorphism of $S$ onto its maximum group homomorphic image $G = S/\sigma$. It is clear that $G$ acts on this quasi-ordered set by automorphisms through multiplication of first coordinates, so $G$ also acts by order automorphisms on the quotient partially ordered set

$$\mathcal{X} = \{ [g, a] : g \in G, a \in E \}.$$ 

$\mathcal{X}$ is down directed and has $\mathcal{Y} = \{ [1, a] : a \in E \}$ as an order ideal isomorphic to $E$ and $S \cong P(G, \mathcal{X}, \mathcal{Y})$. 


In my opinion, this proof is a gem. It is so clear and insightful. That’s not a surprise; it is just a trade mark of all of Douglas Munn’s work. But there is more, this is a totally constructible proof with powerful applications. To see this, I want to turn to the second theorem. It states that inverse semigroups have $E$-unitary covers. But there is no uniqueness to them and the problem arises of how they may be constructed. Norman Reilly and I [26] gave an answer to this question by using the notion of a prehomomorphism from one inverse semigroup to another.

**Definition 2.2.** Let $S$ and $T$ be inverse semigroups. Then a function $\theta: S \rightarrow T$ is a prehomomorphism if $s\theta t \leq s\theta \theta$ for all $s, t \in S$.

It is easy to see that prehomomorphisms preserve inverses and therefore map idempotents to idempotents. We say that $\theta$ is idempotent pure if only idempotents are mapped to idempotents.

Suppose that $G$ is a group. Then Schein [37] has shown that the set $K(G)$ of all cosets of $G$, modulo subgroups of $G$, is a inverse semigroup under the multiplication $*$ where $X*Y$ is the smallest coset that contains the subset $XY$. The semigroup $K(G)$ has many interesting properties. For example, the semilattice of idempotents is anti-isomorphic to the lattice of subgroups of $G$; the units are the one element cosets $\{g\}$, $g \in G$. The zero is $G$. Like the symmetric inverse semigroup, the semigroups $K(G)$ are universal for inverse semigroups in the sense that any inverse semigroup can be embedded in one of the semigroups $K(G)$; [23].

**Theorem 2.3.** Let $S$ be an inverse semigroup and let $\theta: S \rightarrow K(G)$ be an idempotent pure prehomomorphism from $S$ into the inverse semigroup $K(G)$ of all cosets of a group $G$. Then

$$T = \{(s, g) \in S \times G : g \in s\theta\}$$

is an $E$-unitary cover of $S$ with maximum group homomorphic image isomorphic to $H = \{g \in G : s\theta \leq g \text{ for some } s \in S\}$; the fact that $\theta$ is a prehomomorphism implies that $H$ is a subgroup of $G$. (Let us say that $\theta$ is surjective if $H = G$.)

Conversely, each $E$-unitary cover of $S$ with maximum group homomorphic image $G$ is isomorphic to one of this form for some idempotent pure surjective prehomomorphism of $S$ into $K(G)$.

Because $T$ is explicitly given in terms of $\theta$, Munn’s construction can be immediately applied to it and we get the following proposition which must
be classified as folklore. At least, I’ve known about it since the middle 1970’s:

**Proposition 2.1.** Let \( \theta : S \to \mathcal{K}(G) \) be an idempotent pure surjective prehomomorphism. Define a quasi-order \( \preceq \) on \( G \times E \) by

\[
(g, e) \preceq (h, f) \iff \exists s \in S \text{ with } ss^{-1} = e, \ s^{-1}s \leq f \text{ and } h \in g(s\theta).
\]

Then \( T \approx P(G, \mathcal{X}, \mathcal{Y}) \) where \( \mathcal{X} \) is the corresponding partially ordered set and \( \mathcal{Y} = \{[1, e] : e \in E\} \approx E \).

The final result of this section really belongs to the present of the subject and not its history. It is included in this section because it deals directly with \( E \)-unitary inverse semigroups and fits nicely with the historical thread which I have been describing here. As I remarked above, there was something un-natural about the localization restriction in Scheiblich’s description of the free semigroups. Of course, the same kind of complaint can be voiced about the old fashioned description of \( E \)-unitary inverse semigroups because it is an exact analog to Scheiblich’s. We saw that this localization restriction could be removed in the case of the free inverse semigroup.

When this was done we obtained a very natural geometrical interpretation of Munn’s description of the free inverse semigroup by labelled graphs.

Suppose we follow the same pattern in the more general situation. That is, given \( G, \mathcal{X}, \mathcal{Y} \), we consider the triples \((g, x, h) \in G \times \mathcal{X} \times G\) with the idea that this triple corresponds to the pair \((e, k)\) where \( e = g^{-1}x \) and \( k = g^{-1}h\); thus \((g, x, h) = (ge, gk)\). The restrictions \( e, k^{-1}e \in \mathcal{Y} \) require that \( g^{-1}x \) and \( h^{-1}x = h^{-1}ge = k^{-1}e \in \mathcal{Y} \). We see then that two of the triples \((g_1, x_1, h_1)\) and \((g_2, x_2, h_2)\) correspond to the same element \((e, k) \in P(G, \mathcal{X}, \mathcal{Y})\) if and only if

\[
x_1 = g_1e, \ h_1 = g_1k, \ x_2 = g_2e, \ h_2 = g_2k
\]

That is, they both correspond to the same element \((e, k)\) if and only if

\[
g_2 = k'g_1, \ x_2 = k'x_1, \ h_2 = k'h_1
\]

where \( k' = g_2g_1^{-1} = h_2h_1^{-1} \). Thus we get a set in one-to-one correspondence with the elements of \( P(G, \mathcal{X}, \mathcal{Y}) \) if we start with the set of all triples \((g, x, h) \in G \times \mathcal{X} \times G\) such that \( g^{-1}x, h^{-1}x \in \mathcal{Y} \) and define an equivalence relation by

\[
(g_1, x_1, h_1) \sim (g_2, x_2, h_2) \iff \exists k' \in G \text{ with } g_2 = k'g_1, \ x_2 = k'x_1, \ h_2 = k'h_1.
\]
Multiplication of two classes is constructed in such a way as to mimic the multiplication in $P(G, \mathcal{X}, \mathcal{Y})$. This can easily be done because 

$$(g_1, x_1, h_1) \sim (1, g_1^{-1}x_1, g_1^{-1}h_1) \text{ and } (g_2, x_2, h_2) \sim (1, g_2^{-1}x_2, g_2^{-1}h_2)$$

so the product should be

$$(1, g_1^{-1}x_1 \land g_1^{-1}h_1 g_2^{-1}x_2, g_1^{-1}h_1 g_2^{-1}h_2) \sim (kg_1, kx_1 \land x_2, h_2) \text{ where } k = g_2 h_1^{-1}.$$ 

This form of the product has a natural interpretation in terms of the evident partial product on the set of triples $(g, x, h)$ if we consider $x$ as an arrow from the vertex $g$ to the vertex $h$:

$$(a, x, b)(b, z, c) = (a, x \land z, c).$$

For $(g_1, x_1, h_1) \sim (kg_1, kx_1, kh_1) = (kg_1, kx_1, g_2)$ if we take $k = g_2 h_1^{-1}$ and the partial product

$$(kg_1, kx_1, g_2)(g_2, x_2, h_2) = (kg_1, kx_1 \land x_2, x_2).$$

Note that $(kg_1, kx_1, kh_1)$ is the only member of the class of $(g_1, x_1, h_1)$ which ends where $(g_2, x_2, h_2)$ begins. Thus the product is well defined.

Given the triple $(G, \mathcal{X}, \mathcal{Y})$ define $E(G, \mathcal{X}, \mathcal{Y})$ to be the set of all equivalence classes of triples $(g, x, h) \in G \times \mathcal{X} \times G$ such that $g^{-1}x, h^{-1}x \in \mathcal{Y}$ under the multiplication defined above. Then we have the following relativistic structure theorem for $E$-unitary inverse semigroups:

**Theorem 2.4. (Steinberg [42])** Let $\mathcal{X}$ be a partially ordered set, $\mathcal{Y}$ an ideal and subsemilattice of $\mathcal{X}$, and $G$ a group which acts on $\mathcal{X}$ in such a way that $\mathcal{X} = G \mathcal{Y}$. Then $E(G, \mathcal{X}, \mathcal{Y})$ is an $E$-unitary inverse semigroup. Conversely, each $E$-unitary inverse semigroup is isomorphic to $E(G, \mathcal{X}, \mathcal{Y})$ for a unique $G, \mathcal{X}, \mathcal{Y}$ (up to isomorphism and equivalence of group actions).

As a final remark in this section, let me point out that the process involved in Steinberg’s relativity theorem is really a very familiar one. It is one that we expect the students in our beginning linear algebra classes to be completely comfortable with - although often they are not. They come into our classes knowing very well what a vector is - a directed line segment - and how to add directed line segments using the parallelogram law. We mathematicians insist in localizing everything at one particular point (the origin) and we come up with vector spaces. Steinberg’s theorem for $E$-unitary inverse semigroups just reverses the process! Actually, it shows that $E$-unitary inverse semigroups are not really more complicated that vector spaces. The parallelogram law is just the special case where $\mathcal{X} = \mathcal{Y}$ is trivial and $G$ is the group of translations of the plane!
3. The Present: $E^*$-unitary inverse semigroups

Most inverse semigroups have zeros, $E$-unitary ones do not. For this reason Szendrei [44] defined an $E^*$-unitary inverse semigroup to be an inverse semigroup $S$ with zero in which $0 \neq e = e^2 \leq a$ implies $a^2 = a$. Clearly $S$ is $E$-unitary if and only if $S^0$ is $E^*$-unitary and, more generally, any any Rees factor semigroup of an $E$-unitary inverse semigroup is $E^*$-unitary. Later Bullman-Fleming, Fountain, and Gould [2] defined an inverse semigroup $S = S^0$ with zero to be strongly $E^*$-unitary if and only if there is a function $\theta : S \to G^0$ with the following properties

1. $x \theta = 0$ if and only if $x = 0$;
2. $x \theta = 1$ if and only if $x^2 = x$;
3. if $xy \neq 0$ then $(xy)\theta = x\theta y\theta$.

for some group $G$.

In terms of the language introduced earlier, such a function $\theta$ is just a 0-restricted idempotent pure prehomomorphism of $S$ into $G^0$. Further, $\theta$ becomes an idempotent pure prehomomorphism of $S$ into $K(G)$ if we interpret $s\theta$ as the one element coset $\{s\theta\}$ for $s \neq 0$ and 0$\theta$ as $G$, the zero of $K(G)$. Thus $\theta$ gives rise to an $E$-unitary cover of $S$. Since $G$ is the zero of $K(G)$, this prehomomorphism is surjective.

Bullman-Fleming, Fountain, and Gould show that any strongly $E^*$-unitary inverse semigroup is $E^*$-unitary and that any Rees factor semigroup of an $E$-unitary inverse semigroup is strongly $E^*$-unitary. Indeed, if $S = P(G, X, Y)$ is $E$-unitary then the function $\theta : S/I \to G^0$ defined by

$$(e, g)\theta = \begin{cases} g & \text{for } (e, g) \notin I \\ 0 & \text{for } (e, g) \in I \end{cases}$$

is a zero restricted idempotent pure prehomomorphism of $S/I$ into $G^0$.

In fact, as noted independently by Margolis and Steinberg (see [43]) and McAlister [24], the converse is also true. For suppose that $\theta : S \to G^0$ is a 0-restricted idempotent pure prehomomorphism of $S = S^0$ into a group with zero $G^0$. Then

$$T = \{(s, g) \in S \times G : g \in s\theta\}$$

$$= \{(s, g) \in S \times G : g = s\theta \text{ if } s \neq 0\} \cup \{(0, g) : g \in G\}$$

is an $E$-unitary cover for $S$. It is easy to see that $S$ is isomorphic to the Rees factor semigroup $T/I$ with $I = \{(0, g) : g \in G\}$. This means that the structure theorems for $E$-unitary inverse semigroups also apply to strongly $E^*$-unitary inverse semigroups.
Theorem 3.1. Let \( X \) be a partially ordered set with zero, \( Y \) an ideal and subsemilattice of \( X \), and \( G \) a group which acts on \( X \) in such a way that \( X = G \cdot Y \). Let

\[
P^*(G, X, Y) = \{ (e, g) \in Y \times G : g^{-1}e \in Y \text{ and } e \neq 0 \} \cup \{0\}.
\]

Then \( P^*(G, X, Y) \) is a strongly \( E^* \)-unitary inverse semigroup under the multiplication

\[
(e, g)(f, h) = (e \wedge gf, gh) \text{ if } e \wedge gf \neq 0
\]

and other products equal to 0. Each strongly \( E^* \)-unitary inverse semigroup is isomorphic to one of this form for some \( G, X, Y \) as above.

Theorem 3.2. Let \( X \) be a partially ordered set with zero, \( Y \) an ideal and subsemilattice of \( X \), and \( G \) a group which acts on \( X \) in such a way that \( X = G \cdot Y \). Let \( E^*(G, X, Y) \) be the set of equivalence classes of triples \( (g, x, h) \) with \( g^{-1}x, h^{-1}x \in Y \setminus \{0\} \), together with 0 under the multiplication

\[
[g_1, x_1, h_1][g_2, x_2, h_2] = [kg_1, k(g_1 \wedge g_2), h_2] \text{ if } k(g_1 \wedge g_2) \neq 0
\]

where \( k = g_2h_1^{-1} \), and with all other products equal to 0. Then \( E^*(G, X, Y) \) is a strongly \( E^* \)-unitary inverse semigroup. Each strongly \( E^* \)-unitary inverse semigroup is isomorphic to one of this form.

Since the strongly \( E^* \)-unitary inverse semigroup \( P^*(G, X, Y) \) is a Rees factor semigroup of the \( E \)-unitary inverse semigroup \( P(G, X, Y) \), the structure of ideals, maximal subgroups, and congruences can be "read off" from the corresponding results for \( E \)-unitary semigroups. And in a similar way we can get the analogous structures for \( E(G, X, Y) \) and \( E^*(G, X, Y) \) by translating the results for \( P \)-semigroups.

The partially ordered set \( X \) which appears in each of these theorems can be given explicitly, thanks to Munn’s proof of the, so-called, \( P \)-theorem in terms of the idempotent pure prehomomorphism \( \theta : S \to G^0 \). However, neither the group \( G \) nor the prehomomorphism \( \theta \) need be unique. Thus, in distinction to the situation with \( E \)-unitary inverse semigroups we do not have a unique representation for \( E^* \)-unitary inverse semigroups.

The global structure theorem in the form I have described it here has, as I indicated earlier, a natural geometric interpretation. We have a set of vertices \( V (= G) \) and a set of admissible arrows \((u, x, v)\) between the vertices; in the case above \((g, x, h)\) is admissible if both \(g^{-1}x\) and \(h^{-1}x\) belong to \( Y \). The labels on the arrows admit a partial semilattice multiplication, and we have a group \( G \) which acts on both the vertices and edges;
the action on vertices is fix-point free. Because of the partial multiplication on the labels, there is a partial multiplication on arrows which are, in the obvious sense, consecutive. We have an equivalence relation on arrows:

\[(u_1, x_1, v_1) \sim (u_2, x_2, v_2) \iff \exists g \in G \text{ such that } u_2 = gu_1, x_2 = gx_1, v_2 = gv_1\]

and a product is defined on the equivalence classes by translating the first arrow, if possible, so that its end is the beginning of the second one and then taking the equivalence class of product of the consecutive classes, if it exists. Otherwise the product is 0.

This is precisely the situation which pertains to the tiling semigroups which were introduced by Kellendonk [8],[9], and Kellendonk and Lawson [10]; see also Lawson’s book [13]. Lawson [16] has recently axiomatized the structure that is needed to carry out this program. He calls it a mosaic, and the resulting semigroup he calls a mosaic semigroup. I shall not describe in detail the axioms - you should read them in Lawson’s paper or in Steinberg’s [41]. Indeed, Lawson points out that his mosaics a reformulation of what Steinberg calls an abstract geometric representation. As I said above, the ingredients are:

1. a directed graph whose edges are labeled by the non-zero elements of a partially ordered set \( \mathcal{X} \) with zero;
2. a group \( G \) which acts partially by order automorphisms on \( \mathcal{X} \) and in a fix-point free manner on the set \( V \) of vertices of the graph; this gives an obvious action of the group on the graph;
3. a relation, (which for analogy with tiling semigroups, Lawson denotes by \( \in \)), that determines the admissible arrows. The arrow (triple) \((a, x, b)\) is admissible if both \( a \in x \) and \( b \in x \);
4. there are connections between the relation \( \in \) and the order on \( \mathcal{X} \); for example, for each \( a \in A \), the set \( \{x \in \mathcal{X} : a \in x\} \bigcup \{0\} \) is subsemilattice and order ideal of \( \mathcal{X} \).

The set of orbits of admissible arrows, together with zero, is turned into a semigroup under the multiplication:

\[[a, x, b], [c, y, d] = [ga, gx \wedge hy, hd]\]

if there exist \( g, h \in G \) with \( gb = hc \) and \( ga, hd \in gx \wedge hy \neq 0 \), and with all other products equal to zero.

So we have shown that every strongly \( E^* \)-unitary inverse semigroup is a mosaic semigroup. It is shown in Lawson [16] that the converse is also true. Note that proving this requires the existence of a 0-restricted idempotent
pure prehomomorphism from the mosaic semigroup into a group with zero. In order to produce this, Lawson makes use of a construction of Fountain.

A natural question arises. Is every $E^*$-unitary inverse semigroup a strongly $E^*$-unitary inverse semigroup? Frankly, this was my first guess when I started thinking about such semigroups after hearing, I believe at the workshop here six years ago, Gomes and Howie’s structure theorem,[7] for $E^*$-unitary inverse semigroups which are categorical at zero, in terms of Brandt semigroups acting on partially ordered sets. Well, I was right for those semigroups, but not in general. Bullman-Flemming, Fountain and Gould [2] have given a lovely example to show that an inverse semigroup may be $E^*$-unitary without being strongly $E^*$-unitary.

**Example 3.1.** (Bullman-Flemming, Fountain and Gould) Let $\mathcal{Y}$ be the free semilattice on three generators $\alpha, \beta, \gamma$ and form the semilattice of groups $S$ over $\mathcal{Y}$ where each group, except $G_{\alpha\beta\gamma}$ which is $\{0\}$, is infinite cyclic. The linking homomorphism $\theta_{\alpha,a\beta}$ is the squaring function while all the other homomorphisms are identities except those into $G_{\alpha\beta\gamma}$. Thus $S$ has the diagram below:

$$
\begin{array}{ccc}
G_\alpha & G_\beta & G_\gamma \\
\theta & & \\
G_{\alpha\beta} & G_{\alpha\gamma} & G_{\beta\gamma} & G_{\alpha\beta\gamma}
\end{array}
$$

where $\theta$ is the squaring function, and the other maps are identical, or zero. Then $S$ is $E^*$-unitary but not strongly $E^*$-unitary.

**Proof.** Let us denote the generators of the groups $G_\ast$ by $a_\ast$ and suppose that $\phi$ is a 0-restricted prehomomorphism of $S$ into some group $G$. Then $a_\beta a_\beta a_\gamma = a_\beta_3 a_\gamma = a_\gamma a_\beta a_\gamma$ gives $a_\beta \phi = a_\beta a_\gamma \phi = a_\gamma \phi$ and similarly $a_\alpha a_\alpha a_\gamma = a_\gamma a_\alpha a_\gamma$ gives $a_\gamma \phi = a_\alpha a_\gamma \phi = a_\gamma \phi$ so that $a_\alpha \phi = a_\beta \phi = a_\gamma \phi \neq 0$.

But also $a_\alpha a_\alpha a_\beta = a_\beta^3 a_\beta$ while $a_\beta a_\alpha a_\beta = a_\beta^2 a_\beta$ which gives $a_\alpha \phi = a_\beta \phi^2$. Thus $a_\alpha \phi = a_\phi \phi^2$ is idempotent while $a_\alpha$ is not. Hence $\phi$ cannot be idempotent pure and consequently $S$ is not strongly $E^*$-unitary.

\[\square\]
However, it is true that many natural examples of $E^*$-unitary inverse semigroups are strongly $E^*$-unitary; that is, they are Rees factor semigroups of $E$-unitary inverse semigroups.

(1) Tiling semigroups;
(2) $E^*$-unitary inverse semigroups which are categorical at zero; Gomes and Howie [7], and Bulman-Flemming, Fountain, and Gould [2];
(3) almost factorizable $E^*$-unitary inverse semigroups [24];
(4) Toplitz inverse semigroups; these were introduced by Nica [33] and studied by him and by Lawson [15], in his summer school lecture notes.
(5) Inverse semigroups of graphs; these were introduced by Ash and Hall [1]. Lawson devotes a section of his 2001 Summer School lectures to these semigroups; It is interesting that these semigroups, like one-dimensional tiling semigroups of Kellendonk and Lawson which are a special case, can be also shown simply and directly to be strongly $E^*$-unitary by using the relationship between Scheiblich and Munn’s representations of the free inverse semigroup on a set; cf McAlister [25].
(6) Inverse semigroups which are separated over a naturally quasisemilatticed 0-cancellative subsemigroup, which are partially embeddable in groups with zero. Naturally quasisemilatticed semigroups were introduced by McAlister [18] and are related to the semigroups in 5; Examples of such semigroups are Rees factor semigroups of cancellative semigroups whose principal left and right ideals form semilattices under intersection, in particular of free semigroups;
(7) 0-bisimple monoids are $E^*$-unitary if and only if their right unit subsemigroup is cancellative; they are strongly $E^*$-unitary if and only if their right unit subsemigroup can be embedded in a group; [14]. If the semigroup does not have a zero then its right unit subsemigroup obeys Ore’s conditions and so, if it is cancellative, it is embeddable in a group. But if there is a zero this does not apply and we cannot conclude that in this case $E^*$-unitary is equivalent to strongly $E^*$-unitary;

So how does one tell if nice classes on $E^*$-unitary inverse semigroups are strongly $E^*$-unitary without having to treat each situation separately? In fact, one can give an answer to this question in terms of a universal construction.
Let $S$ be an inverse semigroup with zero and denote by $S^* = \{ \bar{s} : s \neq 0 \}$ a set in one-to-one correspondence with the set of non-zero elements of $S$. Now let $G$ be the group $FG(S^*)/N$ where $N$ is the normal subgroup generated by the elements

$$\bar{s}^{-1}\bar{st}\bar{t}^{-1}$$

where $s$, $t$, $st$ are non-zero elements of $S$

and let $\eta$ be the map $S \rightarrow G^0$ defined by

$$s\eta = \begin{cases} N\bar{s} & \text{when } s \neq 0 \\ 0 & \text{when } s = 0 \end{cases}$$

Then $\eta$ is easily seen to be a 0-restricted prehomomorphism of $S$ into $G^0$ and we have the following proposition:

**Proposition 3.1.** Let $\theta$ be a 0-restricted prehomomorphism of $S$ into a group with zero $H^0$. Then there is a unique 0-restricted homomorphism $\phi$ of $G^0$ into $H^0$ such that the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\eta} & G^0 \\
\downarrow{\theta} & & \downarrow{\phi} \\
H^0
\end{array}
$$

commutes. Conversely, if $\phi$ is 0-restricted homomorphism of $G^0$ into $H^0$ then $\theta = \eta\phi$ is a 0-restricted prehomomorphism of $S$ into $H^0$.

**Corollary 3.1.** Let $S$ be an inverse semigroup with zero. Then $S$ is strongly $E^*$-unitary if and only if $\bar{s} \in N$ implies $s^2 = s$.

The criterion in the corollary, though it looks neat, merely shifts the original problem into a problem involving the group $N$; which doesn’t really solve it. The examples above indicate that the answer to this question is closely related to the problem of knowing when a semigroup with zero admits a partial embedding into a group with zero. The following beautiful theorem of Steinberg [43] provides some parameters. More details will be found in Steinberg’s paper elsewhere in these proceedings.

**Theorem 3.3.** Let $C$ be a pseudo variety of groups. Then the following conditions are equivalent:

1. the uniform word problem for $C$ is decidable;
(2) it is decidable whether a finite inverse graph embeds in the Cayley graph of a group in $\mathcal{C}$;
(3) it is decidable whether a finite inverse graph is a Schützenberger graph of an $E$-unitary inverse semigroup with maximum group homomorphic image in $\mathcal{C}$;
(4) it is decidable whether a finite inverse semigroup is a Rees quotient of an $E$-unitary inverse semigroup with a maximum group homomorphic image in $\mathcal{C}$;
(5) it is decidable whether a finite inverse semigroup with zero admits a $0$-restricted idempotent pure prehomomorphism into $G^0$ for some group $G$ in $\mathcal{C}$.

Since the uniform word problem is known to be undecidable for the pseudovariety of all groups, the theorem has the following consequence.

**Theorem 3.4.** It is undecidable whether a finite $E^*$-unitary inverse semigroup is strongly $E^*$-unitary.

With this I will stop. I have only scratched the surface of the topic in this introduction. I would urge you to read the papers of Lawson and Steinberg. There is much of interest to be found in them. And, of course there is the obvious question of what can one say about the structure of $E^*$-unitary inverse semigroups which are not strongly $E^*$-unitary!

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