Infinite dimensional Teichmüller spaces

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Introduction

The aim of these talks is to investigate universal properties of Teichmüller spaces, regardless of the dimension of the Teichmüller space. The majority of Teichmüller theory focusses on Teichmüller spaces of finite type Riemann surfaces. We will look at results which hold for both finite and infinite type cases, or point out where differences arise.

A rough outline is as follows:

- Quasiconformal mappings
- Definition of Teichmüller spaces
- Analytic properties of Teichmüller spaces
- Bi-Lipschitz class of Teichmüller spaces and an application to the finite dimensional setting.

These notes are largely based on the survey articles [2, 3]. See those or the many books on the subject (eg. Ahlfors, Fletcher-Markovic, Gardiner-Lakic, Hubbard, Imayoshi-Taniguchi, Lehto) for further reading.
Quasiconformal mappings

- Recall that conformal mappings send infinitesimal circles to infinitesimal circles. In other words, the derivative of a conformal mapping is a $\mathbb{C}$-linear map.

- **Quasiconformal mappings** are a generalization of conformal mappings which, informally, send infinitesimal circles to infinitesimal ellipses, and where we have a uniform bound on the eccentricity of these ellipses.

- In particular, if we assume a quasiconformal map is differentiable, then its derivative is an $\mathbb{R}$-linear map.

- However, we don’t want to restrict quasiconformal mappings to being differentiable. One of the desired properties we want from quasiconformal mappings is that of compactness, which is not necessarily going to preserve differentiability.
Definition 1

Let $U \subset \mathbb{C}$. A homeomorphism $f : U \rightarrow f(U)$ is called quasiconformal if and only if the following two conditions hold:

(i) (regularity) $f \in W^{1,2}_{2,\text{loc}}(U)$, i.e. $f$ has locally $L^2$-integrable partial derivatives,

(ii) (distortion) $|f_z| \leq k |f_{\bar{z}}|$ almost everywhere in $U$, for some $0 \leq k < 1$.

- We can use other regularity conditions instead of the Sobolev space condition, i.e. that $f$ is absolutely continuous on almost every horizontal and vertical line.
- The regularity condition only implies that $f$ is almost everywhere differentiable in $U$.
- Here $f_z = (f_x - if_y)/2$ and $f_{\bar{z}} = (f_x + if_y)/2$ are the complex derivatives. The distortion condition implies that $f$ is sense-preserving.
Dilatation of a qc mapping

**Definition 2**

If \( f \) is quasiconformal, and (ii) holds for a minimal value \( k = k(f) \), then we say that \( K = \frac{1+k}{1-k} \geq 1 \) is the dilatation (or distortion) of \( f \). We then call \( f \) a \( K \)-quasiconformal mapping.

- If \( f \) is conformal, then the Cauchy-Riemann equations give \( f_z \equiv 0 \), and hence \( f \) is \( 1 \)-quasiconformal. The converse is also true.

- The value \( K \) gives a measure of how far away \( f \) is from a conformal map.

**Example 3**

Let \( K \geq 1 \). Then \( f(x + iy) = Kx + iy \) is easily seen to be \( K \)-quasiconformal.
There are several equivalent ways of defining quasiconformal maps.

- A quadrilateral $Q$ is a Jordan domain with 4 distinguished boundary points. Then $Q$ is conformally equivalent to a unique rectangle $R(0, 1, 1 + Mi, Mi)$, where the distinguished points are mapped to the corners of the rectangle. The module of $Q$ is defined to be $m(Q) = M$. If $f : U \to f(U)$ is a homeomorphism, $f$ is $K$-quasiconformal if and only if for every quadrilateral $Q \subset U$,

$$\frac{m(Q)}{K} \leq m(f(Q)) \leq Km(Q).$$
A homeomorphism \( f : U \to f(U) \) is called \( \eta \)-quasisymmetric if there is an increasing function \( \eta : [0, \infty) \to [0, \infty) \) such that for any \( x, y, z \in U \),

\[
\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq \eta \left( \frac{|x - y|}{|x - z|} \right).
\]

Then quasisymmetry is equivalent to quasiconformality on open sets.

A homeomorphism \( f : U \to f(U) \) is quasiconformal if and only if there is a constant \( H \) such that

\[
\sup_{z \in U} \limsup_{r \to 0} \frac{\sup_{|w - z| = r} |f(z) - f(w)|}{\inf_{|w - z| = r} |f(z) - f(w)|} \leq H.
\]

In fact, \( \limsup \) can be replaced by \( \liminf \) here (Heinonen-Koskela).
We will want to focus on quasiconformal maps $f : \mathbb{H} \to \mathbb{H}$, where $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ is the upper half-plane. If $f : \mathbb{H} \to \mathbb{H}$ is quasiconformal, then $f$ extends to an homeomorphism $\tilde{f} : \mathbb{R} \to \mathbb{R}$. This homeomorphism turns out to be quasisymmetric. Conversely, every quasisymmetric map $\mathbb{R} \to \mathbb{R}$ extends to a quasiconformal mapping from $\mathbb{H}$ to itself. There are many possible quasiconformal extensions of a quasisymmetric map, for example:

- the Beurling-Ahlfors extension is given by an explicit formula:
  \[
  f(x+iy) = \frac{1}{2} \int_0^1 (\tilde{f}(x+ty)+\tilde{f}(x-ty))dt + i \int_0^1 (\tilde{f}(x+ty)-\tilde{f}(x-ty))dt.
  \]

- the Douady-Earle extension (or barycentric extension) is conformally natural, that is, if $f$ is the Douady-Earle extension of $\tilde{f}$, then $A \circ f \circ B$ is the extension of $A \circ \tilde{f} \circ B$, where $A, B$ are Möbius maps which fix $\mathbb{R}$. 

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Infinite dimensional Teichmüller spaces
Beltrami differentials

From our original definition of quasiconformal maps, we have $|f_z| \leq k|f_z|$ almost everywhere.

**Definition 4**

If $f : \mathbb{H} \to \mathbb{H}$ is a quasiconformal map, then the complex dilatation $\mu_f$ of $f$ is $\mu_f = f_\bar{z}/f_z$.

It follows that the complex dilatation of $f$ is in the unit ball $B(\mathbb{H})$ of $L^\infty(\mathbb{H})$. The converse is surprisingly true too. We will call elements in $B(\mathbb{H})$ Beltrami differentials.

**Example 5**

Suppose $f$ is a stretch by factor $K$ in the direction $e^{i\theta}$. If $R_\theta$ denotes rotation by angle $\theta$ and $f_K(x + iy) = Kx + iy$, then $f = R_\theta \circ f_K \circ R_{-\theta}$. Since we can write $f_K(z) = (K + 1)z/2 + (K - 1)\bar{z}/2$, we obtain $f(z) = \frac{(K+1)z}{2} + \frac{e^{2i\theta}(K-1)\bar{z}}{2}$, and hence $\mu_f = e^{2i\theta}(K - 1)/(K + 1)$. 
Let $\mu$ be a Beltrami differential. Then there exists a unique quasiconformal map $f^\mu : \mathbb{H} \to \mathbb{H}$ with complex dilatation $\mu$ and which fixes 0, 1 and $\infty$.

Given $\mu \in B(\mathbb{H})$, the Measurable Riemann Mapping Theorem solves the Beltrami differential equation

$$f_z = \mu f_z.$$ 

There is a one-to-one correspondence

$$\{ \text{normalized qc self maps of } \mathbb{H} \} \leftrightarrow \{ \text{Beltrami differentials} \},$$

where $f \mapsto \mu f$ and $\mu \mapsto f^\mu$. We have that $\mu_{f^\mu} = \mu$ and $f^{\mu_f} = f$. 

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Infinite dimensional Teichmüller spaces
Compositions of qc maps

Quasiconformal maps are not closed under addition.

Example 7
Let \( f(z) = z + \overline{z}/2 \) and \( g(z) = -z \). Then \( f \) is quasiconformal and \( g \) is conformal, but \( (f + g)(z) = \overline{z}/2 \) which is not quasiconformal since it is not sense-preserving.

However, quasiconformal maps are closed under composition. If \( f : \mathbb{H} \to \mathbb{H} \) is \( K \)-quasiconformal and \( g : \mathbb{H} \to \mathbb{H} \) is \( L \)-quasiconformal, then \( f \circ g \) and \( g \circ f \) are both quasiconformal, with dilatation at most \( KL \). By the chain rule, one can calculate that the complex dilatation of the composition is

\[
\mu_{g \circ f} = \frac{\mu_f + r_f(\mu_{g \circ f})}{1 + r_f \mu_f(\mu_{g \circ f})},
\]

where \( r_f = \overline{f_z}/f_z \) has modulus 1. In particular, if \( g \) is conformal, \( \mu_{g \circ f} = \mu_f \), and if \( f \) is conformal, \( \mu_{g \circ f} = r_f(\mu_{g \circ f}) \).
Compactness properties of qc maps

Theorem 8

Let $f_n : \mathbb{H} \to \mathbb{H}$ be a sequence of $K$-quasiconformal maps which converge uniformly on compact subsets to some function $f$. Then either $f$ is a $K$-quasiconformal mapping or a constant.

There are various results on families of $K$-quasiconformal mappings. We will just state Miniowitz’s version of Montel’s Theorem.

Theorem 9

Let $\mathcal{F}$ be a family of $K$-quasiconformal mappings defined on $\mathbb{H}$. If there exist $a, b \in \mathbb{C}$ such that each $f \in \mathcal{F}$ omits $a$ and $b$, then $\mathcal{F}$ is a normal family, i.e. for every sequence $f_n \in \mathcal{F}$, there exists a subsequence which converges uniformly on compact subsets to a $K$-quasiconformal mapping or a constant.
Teichmüller spaces: introduction

We start with the classification of the universal covers of Riemann surfaces.

**Theorem 10 (Uniformization Theorem)**

*Let $S$ be a Riemann surface. The universal cover of $S$ is either $\mathbb{C}$, $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ or $\mathbb{H}$.*

The plane $\mathbb{C}$, punctured plane $\mathbb{C} \setminus \{0\}$ and torus $T$ are the only Riemann surfaces covered by $\mathbb{C}$. The Riemann sphere $\overline{\mathbb{C}}$ is the only Riemann surface covered by $\overline{\mathbb{C}}$. The Teichmüller spaces of $\overline{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$ consist of a single point, while the Teichmüller space of $T$ is biholomorphic to $\mathbb{H}$ and isometric to the hyperbolic plane.

These Teichmüller spaces are well understood and so we will focus on the case where $\mathbb{H}$ is the universal covering.
For future reference, we make the following definitions.

**Definition 11**
We call a Riemann surface *analytically finite* if it is a surface of genus $g$ with $n$ points removed. It is called *analytically infinite* if we allow either $g$ or $n$ to be infinite.

**Definition 12**
A hyperbolic Riemann surface is said to be of *exceptional type* if $2g + n \leq 4$, otherwise it is said to be of *non-exceptional type*. 
We will equip $\mathbb{H}$ with the hyperbolic metric density

$$\rho(z) = \frac{1}{\text{Im}(z)}$$

for $z \in \mathbb{H}$. The universal covering $\pi : \mathbb{H} \rightarrow S$ induces a hyperbolic metric on $S$ and the Riemann surface $S$ is then called hyperbolic.

Let $G$ be a Fuchsian group acting on $\mathbb{H}$ so that $S$ is conformally isometric to $\mathbb{H}/G$. The group $G$ is unique up to conjugation by a Möbius map which fixes $\mathbb{H}$ and is called the covering group. Denote by $\text{PSL}_2(\mathbb{R})$ the subgroup of the Möbius group which fixes $\mathbb{H}$. 
Definition 13

Let $S$ be a hyperbolic Riemann surface with covering group $G$, so that $S$ is isometric to $\mathbb{H}/G$. The Teichmüller space $T(S)$ of $S$ consists of equivalence classes of quasiconformal mappings $f : \mathbb{H} \to \mathbb{H}$ such that

$$f \circ \gamma \circ f^{-1} \in PSL_2(\mathbb{R}),$$

for all $\gamma \in G$. Two such maps $f_1, f_2$ are said to be Teichmüller equivalent if their quasisymmetric extensions to $\mathbb{R}$ agree up to post-composition by a Möbius map, i.e. $f_1 \sim f_2$ if and only if

$$f_1|_{\mathbb{R}} = \beta \circ f_2|_{\mathbb{R}},$$

for some $\beta \in PSL_2(\mathbb{R})$. The base-point of $T(S)$ is the equivalence class of the identity.
The definition of $T(S)$ depends on the Fuchsian group $G$. Since $G$ is unique up to conjugation by an element of $PSL_2(\mathbb{R})$, all subsequent definitions are independent of this choice. We denote by $[f]$ the equivalence class of quasiconformal maps $f : \mathbb{H} \to \mathbb{H}$ satisfying (1), and write $[f] \in T(S)$.

We could work with quasisymmetric maps of $\mathbb{R}$ which satisfy (1) instead of quasiconformal maps of $\mathbb{H}$, thanks to the Douady-Earle extension.

The map $f : \mathbb{H} \to \mathbb{H}$ satisfying (1) projects to a quasiconformal map $\hat{f} : S \to S_1$, where $S \simeq \mathbb{H}/G$ and $S_1$ is a Riemann surface whose covering Fuchsian group is $fGf^{-1}$. The condition in (1) is equivalent to the property that the projections $\hat{f}_1, \hat{f}_2$ map $S$ onto surfaces $S_1$ and $S_2$ respectively and there that $\tilde{f}_2 \circ \tilde{f}_1^{-1} : S_1 \to S_2$ is homotopic to a conformal map $g : S_1 \to S_2$. 
Suppose we just view $S$ as a topological surface. We can put an atlas $(\phi, U)$ on $S$, where $U \subset S$ and $\phi : U \to \mathbb{C}$ is a homeomorphism, by requiring the transition maps to be biholomorphic. Two atlases are called equivalent if their union is also an atlas. An equivalence class of atlases on $S$ is called a complex structure on $S$. Without loss of generality, we may associate an equivalence class of atlases to the maximal atlas.

Two complex structures $(\phi, U)$ and $(\psi, V)$ are called Teichmüller equivalent if there is a homeomorphism $f : S \to S$ that is isotopic to the identity and so that $\phi \circ f = \psi$. The Teichmüller space $T(S)$ can then be defined as the set of Teichmüller equivalence classes of complex structures on $S$. 
Moduli space

- If two complex structures differ just by a homeomorphism (not necessarily isotopic to the identity), then they define the same point in the moduli space $\mathcal{M}(S)$. If this homeomorphism is not isotopic to the identity, then they do not define the same point of $T(S)$, so points of $T(S)$ carry extra information.

- This extra information can be summarized by being an isotopy class of homeomorphisms $f : S \to S$, called a marking of $S$. The act of forgetting the marking defines a map $T(S) \to \mathcal{M}(S)$, which is a universal orbifold covering map.

- The group of homeomorphisms of $S$ isotopic to the identity is denoted $\text{Homeo}_0(S)$ and is a normal subgroup of $\text{Homeo}(S)$. The quotient is called the mapping class group $MC(S)$ and acts on $T(S)$. The resulting quotient $T(S)/MC(S)$ is nothing but $\mathcal{M}(S)$. 
Equivariant Beltrami differentials

Let’s now return to the setting where points in Teichmüller space are equivalence classes of quasiconformal self-mappings of \( \mathbb{H} \).

Suppose that \([f] \in T(S)\) and let \( \mu = \mu_f \) be the Beltrami differential of \( f \). Then \( \|\mu\|_\infty < 1 \) and

\[
\mu(z) = \mu(\gamma(z)) \frac{\gamma'(z)}{\gamma'(\gamma(z))}, \quad z \in \mathbb{H}, \gamma \in G. \tag{2}
\]

In other words, \( \mu \) projects to a \((-1, 1)\)-differential on \( S \). We write \( \mu \in B(G) \) if \( \mu \) satisfies (2). Let \( \mathbb{L} = \mathbb{C} \setminus \mathbb{H} \) be the lower half-plane. Then with \( \mu \) as above, define

\[
\hat{\mu}(z) = \mu(z), \quad z \in \mathbb{H}, \quad \hat{\mu}(z) = 0, \text{ otherwise.}
\]

It follows that \( \hat{\mu} \) satisfies (2) for all \( z \in \mathbb{C} \).
Teichmüller space: Beltrami differentials

By solving the Beltrami differential equation in \( \mathbb{C} \), there exists a quasiconformal map \( f_\mu : \mathbb{C} \to \mathbb{C} \) whose complex dilatation is \( \hat{\mu} \), and \( f_\mu \) is unique if we require it to fix \( 0, 1 \) and \( \infty \). Further, \( f_\mu \) is conformal in \( \mathbb{L} \) and satisfies (1) in \( \mathbb{C} \).

(Note that \( f_\mu \) and \( f_\mu \) need not agree on \( \mathbb{H} \), even though their complex dilatations agree there.)

We can phrase the definition of \( T(S) \) in terms of Beltrami differentials.

**Theorem 14**

Suppose that \( \mu, \nu \in B(\mathbb{H}) \) and satisfy (2). Then the following are equivalent:

(i) \( f_\mu \sim f_\nu \),

(ii) \( f_\mu|_\mathbb{R} = f_\nu|_\mathbb{R} \)

(iii) \( f_\mu|_\mathbb{L} = f_\nu|_\mathbb{L} \).

If any of these hold, we write \( \nu \in [\mu] \in T(S) \).
**Teichmüller metric**

**Definition 15 (Teichmüller metric)**

Let \([f], [g] \in T(S)\). The Teichmüller distance between \([f]\) and \([g]\) is given by

\[
d_T([f], [g]) = \inf_{f_1 \in [f], g_1 \in [g]} \frac{1}{2} \log K(g_1 \circ f_1^{-1}).
\]

This distance is in fact a metric:

- triangle inequality: use the fact that \(K_{f \circ g} \leq K_f K_g\),
- symmetry is more or less obvious,
- \(d_T = 0\) implies we have the same point follows because we can replace \(\inf\) by \(\min\) due to compactness of quasiconformal maps.

In terms of Beltrami differentials,

\[
d_T([\mu], [\nu]) = \inf_{\mu_1 \in [\mu], \nu_1 \in [\nu]} \frac{1}{2} \log \left( \frac{1 + ||\frac{\mu_1 - \nu_1}{1 - \mu_1 \nu_1}||_\infty}{1 - ||\frac{\mu_1 - \nu_1}{1 - \mu_1 \nu_1}||_\infty} \right).
\]
Schwarzian derivatives

We are going to see that $T(S)$ with the Teichmüller metric is a complex Banach manifold. To do this, we first need to construct the Banach spaces we will model $T(S)$ on.

**Definition 16**

Suppose that $U \subset \mathbb{C}$ and that $g$ is a locally injective holomorphic map. Then its **Schwarzian derivative** $S(g)$ is given by

$$S(g) = \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2.$$

If $A$ is a Möbius map, then one can calculate that $S(A) = 0$. The Schwarzian derivative gives a measure of how far $g$ is from a Möbius transformation, i.e. how much the holomorphic map distorts the cross-ratio of four points. We also have the co-cycle property

$$S(g \circ f) = (S(g) \circ f)(f')^2 + S(f).$$
Schwarzians and the Bers space

If we apply the Schwarzian derivative to $f_\mu$ in $\mathbb{L}$, we obtain a holomorphic map $S = S(f_\mu|_\mathbb{L})$ which satisfies

$$S(g(z))g'(z)^2 = S(z), \quad z \in \mathbb{L}, \gamma \in \mathbb{L}.$$ 

where $\rho_\mathbb{L}$ denotes the hyperbolic metric density on $\mathbb{L}$.

**Definition 17 (Bers space)**

Denote by $Q(\mathbb{L}, G)$ the Banach space of all holomorphic maps $\phi : \mathbb{L} \to \mathbb{C}$ which satisfy

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z), \quad z \in \mathbb{L}, \gamma \in G \quad (3)$$

and

$$\|\phi\|_Q := \|\phi\rho_\mathbb{L}^{-2}\|_\infty < \infty,$$

the latter giving the norm on $Q(\mathbb{L}, G)$. 
Schwarzians and the Bers space

- Nehari proved that if $f : \mathbb{D} \to \mathbb{C}$ is conformal, then $\|S(f)\|_Q \leq 6$. The conformal invariance of Schwarzians then yields the fact $S(f_\mu|_L) \in Q(L, G)$.

- We can therefore define the Schwarzian derivative map

$$S : B(G) \to Q(L, G)$$

given by $S(\mu) = S(f_\mu|_L)$.

- If $\phi$ satisfies (3), then $\phi$ projects to a holomorphic quadratic differential on $S$, i.e. a $(2, 0)$-differential. If we denote the space of holomorphic quadratic differentials on $S$ with finite norm

$$\|\hat{\phi}\|_Q = \sup_{z \in S} |\hat{\phi}(z) \rho_S^{-2}(z)|$$

by $Q(S)$ (and observe the expression here is a function on $S$), then there is a bijection between $Q(S)$ and $Q(L, G)$.
Let $S$ be a hyperbolic surface, $S \cong \mathbb{H}/G$. The Schwarzian derivative map $S : B(G) \to Q(\mathbb{L}, G)$ induces an injective map

$$\Phi : T(S) \to Q(\mathbb{L}, G)$$

such that $\Phi(T(S))$ is an open bounded subset of $Q(\mathbb{L}, G)$. The map $\Phi$ is homeomorphic onto its image and defines a global holomorphic chart for $T(S)$.

The statement that $S$ induces $\Phi$ is equivalent to the statement that $\mu_1 \in [\mu]$ if and only if

$$S(f_{\mu_1}|_{\mathbb{L}}) = S(f_{\mu}|_{\mathbb{L}}).$$

This also implies that $\Phi$ is injective.
Theorem 19 (Ahlfors-Weill section)

Given any $\Phi([\mu]) = \phi \in \Phi(T(S)) \subset Q(\mathbb{L}, G)$, there exists a neighbourhood $V_\phi$ of $\phi$ and a holomorphic map $s_\phi : V_\phi \to B(G)$ such that $S \circ s_\phi$ is the identity on $V_\phi$ and $s_\phi \circ S(\mu) = \mu$.

Let $\mu \in B(G)$. Then the quasiconformal map $f^\mu : \mathbb{H} \to \mathbb{H}$ conjugates $G$ onto a Fuchsian group $G^\mu$. Let $S^\mu$ be the Riemann surface isometric to $\mathbb{H}/G^\mu$. There is a natural bijection

$$T(\mu) : T(S^\mu) \to T(S), \quad [g] \mapsto [g \circ f^\mu],$$

which is an isometry for the Teichmüller metric. This $T(\mu)$ is called the translation map.
Let $[\mu] \in T(S)$.

Then $T(\mu)^{-1} : T(S) \to T(S^\mu)$ is the inverse of the translation map sending $[\mu] \in T(S)$ to the base-point in $T(S^\mu)$.

If $\pi : B(G^\mu) \to T(S^\mu)$ is the projection sending a Beltrami differential to its point in Teichmüller space, and $s_{\mu}$ is the Ahlfors-Weill section in a neighbourhood of $0 \in Q(L, G^\mu)$, then $\pi \circ s_{\mu}$ biholomorphically maps a neighbourhood of $0 \in Q(L, G^\mu)$ to a neighbourhood of $0 \in T(S^\mu)$.

Finally, $(\pi \circ s_{\mu})^{-1} \circ T(\mu)^{-1}$ maps a neighbourhood of $[\mu] \in T(S)$ biholomorphically onto a neighbourhood of $0 \in Q(L, G^\mu)$.

This provides a coordinate chart for $T(S)$ near $\mu$, and can be shown to make $T(S)$ into a complex Banach manifold modelled on the various $Q(L, G^\mu)$, so that the Bers embedding is biholomorphic (although only locally bi-Lipschitz).
Tangent space to Teichmüller space

- The Bers embedding \( \Phi : T(S) \to Q(\mathbb{L}, G) \) is a global holomorphic chart for \( T(S) \). Thus the tangent space at the base-point of \( T(S) \) is identified with \( Q(\mathbb{L}, G) \).

- Let \([\mu] \in T(S)\). Then the translation map \( T(\mu) : T(S^\mu) \to T(S) \) is biholomorphic and maps the base-point in \( T(S^\mu) \) to \([\mu] \in T(S)\). Thus the tangent space at \([\mu] \in T(S)\) is isomorphic to the tangent space at the base-point of \( T(S^\mu) \), i.e. \( Q(\mathbb{L}, G^\mu) \).

- Since the Schwarzian derivative map \( S : B(G) \to Q(\mathbb{L}, G) \) is holomorphic, a differentiable path \( t \mapsto \mu_t \) in \( B(G) \) projects to a differentiable path \( t \mapsto S(\mu_t) \) in \( Q(\mathbb{L}, G) \). Conversely, the Ahlfors-Weill section shows a differentiable path in a neighbourhood of \( 0 \in Q(\mathbb{L}, G) \) lifts to a differentiable path in \( B(G) \) through 0.
Tangent class of Beltrami differentials

Since the derivative of a differentiable path in $B(G)$ gives an element of $L^\infty(G)$, each Beltrami differential $\mu \in L^\infty(G)$ represents a tangent vector at the base-point in $T(S)$, and conversely each tangent vector at the base-point of $T(S)$ is represented by some $\mu \in L^\infty(G)$.

A single tangent vector is represented by many Beltrami differentials. We denote by $[\mu]_{\text{tan}}$ the class of all Beltrami differentials representing the same tangent vector as $\mu \in L^\infty(G)$.

We want to determine how to characterize $[\mu]_{\text{tan}}$. Let $S \simeq \mathbb{H}/G$ and $\omega$ be a fundamental polygon for $G$ in $\mathbb{H}$. Let

$$A^1(G) = \{ \phi : (\phi \circ \gamma)(\gamma')^2 = \phi, \gamma \in G, \|\phi\|_{L^1(\omega)} = \int_{\omega} |\phi| < \infty \}$$

be the Bergman space of holomorphic quadratic differentials which are $L^1$-integrable.
Determining $[\mu]_{\tan}$

**Theorem 20 (Ahlfors-Bers)**

The Schwarzian derivative map $S : B(G) \to T(S)$ has a Fréchet derivative $S' = P$ which is a bounded linear projection map $P : L^\infty(G) \to Q(\mathbb{L}, G)$ given by

$$P(\mu)(z) = -\frac{6}{\pi} \int_{\mathbb{H}} \frac{\mu(w)}{(w - \overline{z})^4} dw.$$  

The kernel of $P$ is

$$N(G) = \{ \mu \in L^\infty(G) : \int_G \mu \phi = 0, \phi \in A^1(G) \}.$$  

This implies $P$ induces a linear isomorphism

$$\hat{P} : L^\infty(G)/N(G) \to Q(\mathbb{L}, G),$$

and that $[\mu]_{\tan} = \mu + N(G)$. 

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The Bergman space

The space $A^1(G)$ is called the Bergman space. It can be identified with the space $A^1(S)$ of holomorphic quadratic differentials on $S$ with finite $L^1$-norm.

Theorem 21

*The dual of $Q_0(\mathbb{H}, G)$ can be identified with $A^1(G)$. The dual of $A^1(G)$ can be identified with $Q(\mathbb{H}, G)$.*

In fact, every linear functional $L$ on $A^1(G)$ has the form

$$L(\phi) = \int_\omega \rho_{\mathbb{H}}^{-2} \overline{\psi} \phi,$$

for some $\psi \in Q(\mathbb{H}, G)$.

Theorem 22

*If $S$ is of finite analytic type, then $Q_0(\mathbb{H}, G)$, $A^1(G)$ and $Q(\mathbb{H}, G)$ all coincide and have finite complex dimension $3g - 3 + n$.***
Recall the Teichmüller distance between \([f]\) and \([id]\) is

\[
d_T([f], [id]) = \inf_{f_1 \in [f]} \frac{1}{2} \log K(f_1), \quad [f] \in T(S).
\]

Since the family of normalized \(K\)-quasiconformal mappings is compact with respect to the topology of local uniform convergence, the infimum is achieved for some map \(f_* \in [f]\). Such a mapping \(f_*\) is called extremal.

An extremal map has the smallest dilatation among all maps homotopic to \(f\), and is not necessarily unique.

**Example 23 (Strebel’s chimney)**

Let \(\Omega\) be the \(\mathbb{L}\) union the half-strip \((0, 1) \times [0, \infty)\). Then the affine stretch \(f(x + iy) = x + iKy\) is extremal for its boundary values. However, if \(f_1\) agrees with \(f\) in the chimney and is the identity in \(\mathbb{L}\) then \(f_1\) is also extremal.
A Beltrami differential $\mu$ is called extremal in $[\mu]$ if the corresponding quasiconformal map is extremal.

**Theorem 24**

*If $\mu$ is extremal, then*

$$
\mu_t = \frac{(1 + |\mu|)^t - (1 - |\mu|)^t}{(1 + |\mu|)^t + (1 - |\mu|)^t} \cdot \frac{\mu}{|\mu|},
$$

*for $0 \leq t \leq 1$ is extremal for $[\mu_t]$. The arc $t \mapsto [\mu_t]$ is a geodesic from 0 to $[\mu_t]$ and $d_T([\mu_t], [0]) = td_T([\mu], [0])$.*

We emphasize that an extremal mapping gives us a geodesic in Teichmüller space, and a uniquely extremal mapping implies there is one geodesic between a given point and the base-point. If there is more than one extremal mapping, then there may be more than one geodesic connecting points of Teichmüller space.
Which mappings are extremal?

**Definition 25**

Let \( \phi \in A^1(G) \). The Beltrami differential \( k|\phi|/\phi \), for \(-1 < k < 1\), is said to be of Teichmüller type. A quasiconformal mapping whose Beltrami differential is of Teichmüller type is called a Teichmüller map.

The following is a famous result of Teichmüller.

**Theorem 26 (Teichmüller’s Theorem)**

*Let \( S, S_1 \) be closed hyperbolic Riemann surfaces and \( f : S \to S_1 \) a quasiconformal map. Then \([f]\) contains a unique extremal map which is a Teichmüller map.*

Hence any two points in \( T(S) \), for \( S \) closed, are connected by a unique geodesic and \( T(S) \) is homeomorphic to the unit ball in the Euclidean space \( \mathbb{R}^{6g-6} \).
To extend such results to arbitrary Riemann surfaces, the following result is key.

**Theorem 27 (Reich-Strebel inequality)**

Let $S \cong \mathbb{H}/G$, $\omega$ be a fundamental polygon for $G$ and let $\mu \in [0]$. Then

$$
\int_{\omega} |\phi| \leq \int_{\omega} \frac{1 + \mu \phi/|\phi|^2}{1 - |\mu|^2} |\phi|,
$$

for all $\phi \in A^1(G)$.

If $g \in [f]$, we can apply the Reich-Strebel inequality to $\mu(f \circ g^{-1})$. If $\mu_f$ is of Teichmüller type, then the Reich-Strebel inequality and the chain rule give the uniqueness part of the Teichmüller theorem.
Strebel gave an example of a homotopy class of a quasiconformal mapping of $\mathbb{H}$ which does not contain a Teichmüller mapping. He did however give sufficient conditions for a class to contain one. Let $C \subset S$ be compact and denote by $\tilde{C}$ the lift to $\mathbb{H}$. Define $H_{\tilde{C}}(f) = \|K_z(f)_{\mathbb{H}\setminus\tilde{C}}\|_\infty$, the boundary dilatation.

**Theorem 28 (Frame Mapping Condition)**

Let $S$ be an infinite type Riemann surface and let $[f] \in T(S)$. Let $K_0$ be the dilatation of an extremal map $f_0 \in [f]$. If there is a compact set $C \subset S$ and $f_1 \in [f]$ such that $H_{\tilde{C}}(f_1) < K_0$, then the homotopy class $[f]$ of $f$ contains a Teichmüller map.

The mapping $f_1$ is called a **frame mapping**. A point $[f] \in T(S)$ is called a **Strebel point** if it contains a frame mapping. The set of Strebel points in $T(S)$ is dense and open (Earle-Zi and Lakic).
A Beltrami differential $\mu_0 \in [\mu]_{tan}$ is called infinitesimally extremal if and only if $||\mu_0||_\infty \leq ||\nu||_\infty$ for any $\nu \in [\mu]_{tan}$.

Theorem 29

*The Beltrami differential $\mu$ is extremal in $[\mu]$ if and only if $\mu$ is infinitesimally extremal in $[\mu]_{tan}$.*

This result is useful because the tangent equivalence relation is linear, whereas the Teichmüller relation is not. The proof follows from the Reich-Strebel Inequality. The notion of unique extremality has the corresponding definitions in the Teichmüller class and the infinitesimal class.
Generalized Teichmüller type

- By Teichmüller’s Theorem, a Teichmüller mapping is uniquely extremal. Strebel gave an example of an extremal map which is not a Teichmüller map (and the example is not uniquely extremal).

- Further, Strebel showed that the horizontal stretching in an infinite strip is uniquely extremal. The Beltrami differential of this map is of the form $k|\phi|/\phi$, but the holomorphic quadratic differential $\phi$ is not integrable. Hence by the definition it is not a Teichmüller Beltrami differential.

- Such objects are called generalized Teichmüller Beltrami differentials and Strebel showed that these are sometimes uniquely extremal, but may not always be.
The Beltrami differential $\mu$ is uniquely extremal in $[\mu]$ if and only if it is infinitesimally uniquely extremal in $[\mu\tan]$. 

Let $\phi \in Q(\mathbb{H}, G)$, $\phi$ not identically zero and let $f : \mathbb{H} \to \mathbb{H}$ be quasiconformal with $\mu_f = k|\phi|/\phi$ for some $0 < k < 1$. Then $f$ is uniquely extremal if and only if there exists a sequence $\phi_n \in A^1(G)$ with

1. $\phi_n \to \phi$ uniformly on compact subsets of $\mathbb{H}$,
2. $k\|\phi_n\|_{L^1(\omega)} - Re \int_\omega \phi_n \mu \to 0$ as $n \to \infty$.

Such a sequence is called a Hamilton sequence. Moreover, BLMM show that not every uniquely extremal map is a generalized Teichmüller map, and moreover need not have constant absolute value.
Uniqueness of geodesics

We have already seen that if $\mu_* \in [\mu]$ is extremal, then $t \mapsto [t\mu_*]$ for $0 \leq t \leq 1$ is a geodesic connecting $[0]$ with $[\mu]$.

Theorem 32 (Earle-Kra-Krushkal)

Let $S \cong \mathbb{H}/G$, $\mu$ an extremal Beltrami differential on $S$ with $[\mu] \neq [0]$. Then the following are equivalent:

(i) $\mu$ is uniquely extremal and $|\mu| = \|\mu\|_\infty$ almost everywhere,

(ii) there is exactly one geodesic segment connecting $[0]$ and $[\mu]$,

(iii) there is exactly one holomorphic isometry $\Psi : \mathbb{D} \to T(S)$ such that $\Psi(0) = [0]$ and $\Psi(\|\mu\|_\infty) = [\mu]$,

(iv) there is exactly one holomorphic isometry $\hat{\Psi} : \mathbb{D} \to B(G)$ such that $\hat{\Psi}(0) = 0$ and $\hat{\Psi}(\|\mu\|_\infty) = \mu$.

Compare this with the results of BLMM.
Li (1991,1993): in any infinite dimensional Teichmüller space, there are pairs of points with infinitely many Teichmüller geodesics connecting them.

Fan (2011): in any asymptotic Teichmüller space there exist pairs of points with infinitely many geodesics connecting them.

Fan (2011): there exists a pair of points in $T(\mathbb{H})$ connected by only one geodesic, but there is more than one geodesic connecting them in their projections to $AT(\mathbb{H})$. 
Definition 33 (Asymptotically conformal)

A quasiconformal map $f : \mathbb{H}/G \to \mathbb{H}/G$ is called asymptotically conformal if for all $\epsilon > 0$ there exists a compact set $K \subset \mathbb{H}/G$ such that $\|\mu_f|_{(\mathbb{H}/G)\setminus K}\|_\infty < \epsilon$.

We also say that the complex dilatation of $f$ vanishes at infinity.

Definition 34 (Asymptotic Teichmüller space)

Define the equivalence relation $\sim_{\text{AT}}$ on $T(S)$ by $[f] \sim_{\text{AT}} [g]$ if and only if $[f \circ g^{-1}]$ contains an asymptotically conformal representative. Then asymptotic Teichmüller space is $AT(S) = T(S)/\sim_{\text{AT}}$. 
Asymptotic Bers map

The Bers map $\Phi : T(S) \to Q(\mathbb{L}, G)$ induces the asymptotic Bers map

$$\hat{\Phi} : AT(S) \to Q(\mathbb{L}, G)/Q_0(\mathbb{L}, G),$$

where $Q_0(\mathbb{L}, G)$ is the subspace of $Q(\mathbb{L}, G)$ consisting of those $\psi$ which vanish at infinity in the sense that for all $\epsilon > 0$ there exists a compact set $K \subset \mathbb{L}/G$ with

$$\left\| \psi \rho_{\mathbb{L}}^{-2} \right\|_{(\mathbb{L}/G)\setminus K} \|_\infty < \epsilon.$$

Earle-Gardiner-Lakic proved that $\hat{\Psi}$ is a local homeomorphism. Earle-Markovic-Saric proved that $\hat{\Psi}$ is a biholomorphic map onto a bounded open subset of $Q(\mathbb{L}, G)/Q_0(\mathbb{L}, G)$. 
The aim is to classify all biholomorphic maps between Teichmüller spaces, and in particular all biholomorphic automorphisms of $T(S)$.

Recall that the mapping class group $MC(S)$ acts on $T(S)$ by $[f] \mapsto [f \circ g^{-1}]$ for $[g] \in MC(S)$ and $[f] \in T(S)$. Hence any $[g] \in MC(S)$ induces a biholomorphic map (in fact an isometry) of $T(S)$ onto itself. Such biholomorphic maps are called geometric.

More generally, a biholomorphic map $T(S) \to T(S_1)$ induced by a quasiconformal map from $S_1$ onto $S$ is said to be geometric.

Are these the only biholomorphic maps of Teichmüller space? Royden proved this was so for closed surfaces. The first step is proving equality of Kobayashi and Teichmüller metrics.
The Kobayashi pseudo-metric

Let $X$ be any connected complex Banach manifold and let $H(D, X)$ be the set of all holomorphic maps from $\mathbb{D}$ into $X$. The Kobayashi function is

$$\delta_X(x, y) = \inf\{\rho_D(0, t) : f(0) = x, f(t) = y, f \in H(D, X)\},$$

provided the set of such $f$ is non-empty, and $+\infty$ otherwise. If $f : X \to Y$ is holomorphic, then

$$\delta_Y(f(x_1), f(x_2)) \leq \delta_X(x_1, x_2), \quad x_1, x_2 \in X$$

with equality if $f$ is biholomorphic.

Definition 35 (Kobayashi pseudo-metric)

The Kobayashi pseudo-metric $\sigma_X$ on $X$ is the largest pseudo-metric on $X$ such that

$$\sigma_X(x, y) \leq \delta_X(x, y), \quad x, y \in X.$$  

If $\delta_X$ is a metric, then $\sigma_X = \delta_X$. 
The Kobayashi pseudo-metric is the largest pseudo-metric for which holomorphic mappings are contractions. If $S$ is a hyperbolic Riemann surface, then the Kobayashi metric coincides with the hyperbolic metric on $S$. We remark that the Kobayashi pseudo-metric is rarely Riemannian, but does have a Finsler structure on $T(S)$.

**Theorem 36**

*Let $S$ be a hyperbolic Riemann surface. Then the Teichmüller metric and Kobayashi metric coincide on $T(S)$.*

This theorem is due to Royden for closed Riemann surfaces and Gardiner for geometrically infinite Riemann surfaces. Earle-Kra-Krushkal gave a unified proof of Royden’s Theorem via the theory of holomorphic motions.
Royden’s Theorem

**Theorem 37 (Royden)**

*If $S$ is a non-exceptional closed surface, then each biholomorphic self-map of $T(S)$ is geometric.*

We will outline the proof of this theorem.

1. The biholomorphic map $\psi : T(S) \rightarrow T(S)$ is an isometry for the Kobayashi metric because it is biholomorphic. Hence it is also an isometry for the Teichmüller metric.
2. The derivative map $\psi' : T_{[id]}(T(S)) \rightarrow T_{\psi([id])}(T(S))$ gives an isometry between the tangent spaces at the base-point in $T(S)$ and its image.
3. We have seen that the tangent space at the base-point is isometric to the dual of the space of all integrable holomorphic quadratic differentials on $S$. Since the tangent space is finite dimensional, an isometry between tangent spaces gives an isometry between the pre-duals.
Proof of Royden’s Theorem

• We have an induced isometry between $A^1(S)$ and the space of integrable holomorphic quadratic differentials on $\psi([id])(S)$.

• Suppose $S \simeq \mathbb{H}/G$, $\psi([id]) = [f]$ and let $G_f = fGf^{-1}$ be the conjugate Fuchsian group. We have an induced linear isometry $L : A^1(G) \to A^1(G_f)$.

Definition 38

An isometry $L : A^1(G) \to A^1(G_1)$ is called geometric if it can be written

$$L(\phi) = \theta(\phi \circ \alpha)(\alpha')^2, \quad \phi \in A^1(G),$$

for some $|\theta| = 1$, where $\alpha \in PSL_2(\mathbb{R})$ induces a conformal map $\tilde{\alpha} : \mathbb{H}/G_1 \to \mathbb{H}/G$.

• The key part of Royden’s proof is showing that a linear isometry between $\mathbb{H}/G$ and $\mathbb{H}/G_f$ is geometric whenever $\mathbb{H}/G$ is a non-exceptional closed surface.
Proof of Royden’s Theorem

- We therefore have a conformal map between \( \mathbb{H}/G \) and \( \mathbb{H}/G_f \) which in turn implies the basepoint \([id] \in T(S)\) is mapped by an element \(\rho_{[id]} \in MC(S)\) onto \([f] \in T(S)\). Similarly, \([g] \in T(S)\) is mapped by an element \(\rho_{[g]} \in MC(S)\) onto \(\Psi([g]) \in T(S)\). Since \(MC(S)\) acts properly discontinuously on \(T(S)\), \(\rho\) is independent of \(g\), and it follows that \(\Psi = \rho\). This completes the sketch proof.

- Earle-Kra, Earle-Gardiner and Lakic all extended Royden’s Theorem to cover wider and wider classes of Riemann surfaces using the Royden machinery.

- The full proof of Royden’s Theorem for all Riemann surfaces of geometrically infinite type was given by Markovic, who showed that an isometry between Bergman spaces of two Riemann surfaces is necessarily geometric (along the way reproving a result of Rudin).
Theorem 39

Let $S$ be a Riemann surface of non-exceptional type. Then $\text{Aut}(T(S))$ coincides with $\text{MC}(S)$. If there is a biholomorphic map $T(S) \rightarrow T(S_1)$, then $S, S_1$ must be quasiconformally related.

This circle of ideas shows that a linear isometry between Bergman spaces of two Riemann surfaces implies the surfaces are in fact conformally equivalent. We will now investigate a counterpart to this, by considering what happens if the isometry is weakened to an bi-Lipschitz isomorphism.
Theorem 40 (Fletcher)

Let $S$ be a Riemann surface of infinite type. Then $A^1(S)$ is isomorphic to the sequence space $\ell^1(\mathbb{C})$.

- The conclusion here is that there is an isomorphism $\alpha_S : A^1(S) \rightarrow \ell^1$ and a constant $C_S$ such that
  $$\frac{\|\alpha_S(\phi)\|_{\ell^1}}{C_S} \leq \|\phi\|_1 \leq C_S\|\alpha_S(\phi)\|_{\ell^1},$$
  for all $\phi \in A^1(S)$.
- If $S, S_1$ are Riemann surfaces of infinite type, then $A^1(S)$ and $A^1(S_1)$ are isomorphic. Note that $S, S_1$ need not even be homeomorphic here. Recall that if we have an isometry, then the surfaces are quasiconformally equivalent.
- The proof uses a decomposition of $S$ into small pieces, certain projections and a classical functional analysis result of Pelczynski.
Consequences of the bi-Lipschitz class of $A^1(S)$

**Corollary 41**

If $S \simeq \mathbb{H}/G$ is of infinite type, then $Q(L, G)$ is isomorphic to $\ell^\infty$ and $Q_0(L, G)$ is isomorphic to $c_0 \subset \ell^\infty$.

Combining this corollary with the Bers embedding (which is a locally bi-Lipschitz map), we have the following theorem.

**Theorem 42 (Fletcher)**

If $S, S_1$ are Riemann surfaces of infinite type, then $T(S)$ and $T(S_1)$ (respectively $AT(S)$ and $AT(S_1)$) are locally bi-Lipschitz equivalent.

Up to bi-Lipschitz maps, the Teichmüller space of any infinite type surface always looks the same.
The Carathédory metric is the smallest metric on a connected complex Banach manifold for which holomorphic maps are distance decreasing. Is the Carathéodory metric equal to the Teichmüller metric on $T(S)$? Yes, if $S = \mathbb{H}$ by results of Krushkal. See also results of Kra on abelian Teichmüller disks.

There is an isomorphism $\alpha_S : A^1(S) \rightarrow \ell^1$ if $S$ is of infinite type. Is there a universal constant $C$ so that $\alpha_S$ is always $C$-bi-Lipschitz?

Is the Teichmüller metric on $AT(S)$ equal to the Kobayashi metric? Can one characterize biholomorphic maps of $AT(S)$?
The covering number

The rest of these slides are on results of Fletcher-Kahn-Markovic [1].

**Question 43**

*Given a metric space* \((X, d)\), a set \(E \subset X\) and \(r > 0\), how many balls of radius \(r\) does it take to cover \(E\)?

**Definition 44**

The *\(r\)-covering number* \(\eta_X(E, r)\) is the minimal such number.

**Example 45**

If \(X = \mathbb{R}^n\), \(d\) is the supremum norm and \(E = (0, R)\), then 
\[
\eta(E, r) \sim (R/r)^n.
\]

Note that \(\eta\) may be infinite, e.g. \((X, d) = \ell^\infty\), \(E = B(0, 1)\) and \(r < 1\).
The covering number

- If $E, F$ are subsets of $X$ with $E \subset F$, then $\eta(E, r) \leq \eta(F, r)$.
- If $Y$ is a subspace of $X$ and $E \subset Y$, then
  $$\eta_X(E, r) \leq \eta_Y(E, r) \leq \eta_X(E, r/2).$$
- If $(X, d')$ is a real $n$-dimensional Banach space, then for $0 < r < R$, we have $\eta(B(0, R), r) \geq (R/r)^n$.
- If $f : (X, d_X) \to (Y, d_Y)$ is an $L$-bi-Lipschitz map, then for $E \subset X$ and $r > 0$, we have $\eta_X(E, r) \leq \eta_Y(f(E), r/L)$.

This last property we call the quasi-invariance of the covering number under bi-Lipschitz maps.
The main question

Let \( g \geq 2 \). Fix a closed surface \( S_g \) of genus \( g \), then Teichmüller space is

\[
\mathcal{T}_g = \{ (S, f) : f : S_g \to S \text{ a homeomorphism, } S \text{ a Riemann surface} \}.
\]

Usually we will suppress the marking \( f \) in notation. Moduli space \( \mathcal{M}_g \) is the quotient by the mapping class group. For \( \epsilon > 0 \), denote by \( \mathcal{M}_{g,\epsilon} \), the thick part of moduli space. The Teichmüller metric \( d_T \) on \( \mathcal{T}_g \) is

\[
d_T(S_1, S_2) = \inf \{ \log \sqrt{K} : f : S_1 \to S_2 \text{ is } K \text{ – quasiconformal} \},
\]

and is defined on moduli space as the quotient metric.

Question 46

Estimate \( \eta \) in the setting of \( \mathcal{M}_{g,\epsilon} \) and \( \mathcal{T}_g \), where both are equipped with the Teichmüller metric.
The main results

**Theorem 47 (Fletcher-Kahn-Markovic)**

Let $\epsilon > 0$ and $r > 0$. Then there exist constants $c_1, c_2$ depending only on $\epsilon$ and $r$ such that

$$(c_1 g)^{2g} \leq \eta(M_{g,\epsilon}, r) \leq (c_2 g)^{2g},$$

for all large $g$.

**Theorem 48 (Fletcher-Kahn-Markovic)**

Let $S \in T_g$ be a Riemann surface of injectivity radius $\epsilon > 0$, $R > 0$ and $B = B_{T_g}(S, R)$ be a ball in Teichmüller space. Then there exist constants $d_1 = d_1(R, r) > 0$ and $d_2 = d_2(\epsilon, R, r) > 1$ such that

$$d_1^g \leq \eta(B, r) \leq d_2^g,$$

for all large $g$, and where $d_1 \to \infty$ as $r \to 0$ for any fixed $R > 0$.
To prove Theorem 47, we take the following steps:

(i) Let $\epsilon > 0$, then there exist $r_u > 0$ and $c_u > 0$ such that

$$\eta(M_{g,\epsilon}, r_u) \leq (c_u g)^{2g},$$

for all large $g$.

(ii) Let $\epsilon > 0$, then there exist $r_\ell > 0$ and $c_\ell > 0$ such that

$$\eta(M_{g,\epsilon}, r_\ell) \geq (c_\ell g)^{2g},$$

for all large $g$.

(iii) Prove Theorem 48.

Using Theorem 48 and then quotienting to $M_g$, we can extend the inequalities from Steps (i) and (ii) to all values of $r$. 

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Infinite dimensional Teichmüller spaces
Step (i), triangulations

**Definition 49**

A genus $g$ triangulation is a pair $(\tau, \iota)$, where $\tau$ is a connected graph and $\iota : \tau \to S_g$ is an embedding such that every component of $S_g \setminus \iota(\tau)$ is a topological disk bounded by three edges from $\iota(\tau)$.

**Definition 50**

We call two genus $g$ triangulations $(\tau_i, \iota_i)$ for $i = 1, 2$ equivalent if there is a homeomorphism $h : S_g \to S_g$ such that $h(\iota_1(\tau_1)) = \iota_2(\tau_2)$ and $h$ maps vertices and edges to vertices and edges respectively.

**Definition 51**

Denote by $\Delta(k, g)$ the set of genus $g$ triangulations such that each vertex of $\tau$ has degree at most $k$ and $\tau$ has at most $kg$ vertices and edges.
Step (i), counting triangulations

- If two triangulations in $\Delta(k, g)$ are equivalent, then there exists $K_0 = K_0(\epsilon)$ and a $K_0$-quasiconformal map sending one triangulation to the other.

- If $S$ is a Riemann surface of genus $g \geq 2$ with injectivity radius at least $\epsilon > 0$, then there exists a triangulation in $\Delta(k, g)$ where each edge of $\nu(\tau)$ has length between $\epsilon/2$ and $\epsilon$.

- For each equivalence class of triangulations, choose a representative $S_i$. Then if $K > K_0$, the collection $B(S_i, \log \sqrt{K})$ covers $\mathcal{M}_{g, \epsilon}$.

- A result of Kahn and Markovic counts the number of equivalence classes of triangulations in $\Delta(k, g)$, which gives us Step (i).
Step (ii), the lower bound

- Fix a base Riemann surface $S_0$ of genus 2 with injectivity radius $\epsilon$, and fix a triangulation $(\tau_0, \iota_0)$ of $S_0$ with vertices $v_1, \ldots, v_n$ and control on the lengths of the edges in terms of $\epsilon$.

- Consider the unbranched genus $g$ covers of $S_0$. By a result of Muller and Puchta, there exists a constant $P > 0$ such that the number of such covers is at least $(Pg)^{2g}$ for large $g$.

- Let $S_1$ be a cover of $S_0$ in $\mathcal{M}_{g,\epsilon}$ and let $(\tau_1, \iota_1)$ be a lift of the triangulation $(\tau_0, \iota_0)$ to $S_1$. Since the degree of such a cover is $g - 1$, each $v_i$ has $g - 1$ pre-images in $S_1$, and denote the set of such pre-images by $W_{i}^{S_1}$ for $i = 1, \ldots, n$. 

Step (ii), the lower bound

- If $S_2$ is another such cover with corresponding triangulation $(\tau_2, \iota_2)$ and $f : S_1 \to S_2$ is a quasiconformal map with small enough maximal dilatation which maps $W_{i}^{S_1}$ to $W_{i}^{S_2}$ for each $i = 1, \ldots, n$, then $S_1$ and $S_2$ are actually conformally equivalent.

- Next, make a fine grid on $S_1$ with respect to the triangulation $(\tau_1, \iota_1)$. The triangulation $(\tau_2, \iota_2)$ on $S_2$ pulls back to $S_1$ and we can then perturb it so that its vertices coincide with the grid.

- Label each element of the grid with $i$ if the corresponding vertex is in $W_{i}^{S_2}$. In this way, we can associate a labelling of the grid to each $S \in \mathcal{M}_{g,\epsilon}$ and by the above observation, if two elements give the same labelling, they give the same point in moduli space.

- By estimating the number of labellings of the grid, and combining with the Muller-Puchta estimate, we obtain the lower bound.
Step (iii), the covering number for $T_g$

Teichmüller space is a complicated metric space, and so the general strategy for estimating $\eta$ is to reduce to a simpler metric space and use the quasi-invariance of the covering number under bi-Lipschitz mappings. We do this in two steps:

(a) use the Bers embedding of Teichmüller space into the linear Banach space $Q(S)$;

(b) construct a bi-Lipschitz map from $Q(S)$ into $\mathbb{C}^n$, with control over the dimensions $n$ and the isometric distortion of the bi-Lipschitz map.

Recall that for a Riemann surface $S$, the Bers space $Q(S)$ is the Banach space of holomorphic quadratic differentials on $S$ with the hyperbolic-supremum norm

$$\|\varphi\|_Q = \sup_{z \in S} \rho_S^{-2}(z)|\varphi(z)|,$$

where $\rho_S$ is the hyperbolic density on $S$. 
Step (iii), the Bers embedding

- Once we fix a base-point $S \in \mathcal{T}_g$, the Bers embedding $\mathcal{T}_g \to Q(S)$ is a holomorphic map which sends $S$ to 0.
- The Bers embedding is locally bi-Lipschitz, and its image is contained in the ball $B_Q(0, 6)$ in $Q(S)$.
- Given $R > 0$, there exist constants $a = a(R)$ and $L = L(R)$ such that

$$\eta_Q(B_Q(0, a), 2Lr) \leq \eta_{\mathcal{T}_g}(B_{\mathcal{T}_g}(S, R), r) \leq \eta_Q(B_Q(0, 6), r/2L).$$

- Since $Q(S)$ is a real Banach space of dimension $6g - 6$, the left hand inequality gives the lower bound in Theorem 48. For the upper bound, we need to estimate $\eta_Q$. 
Theorem 52 (Fletcher)

Let $S_1, S_2$ be Riemann surfaces of infinite analytic type (e.g. the plane punctured at lattice points, or an infinite doughnut; in particular they need not be homeomorphic). Then $Q(S_1)$ is isomorphic to $\ell^\infty$ (and hence also to $Q(S_2)$), and Teichmüller space $\mathcal{T}(S_1)$ is locally bi-Lipschitz to $\mathcal{T}(S_2)$.

We want a finite dimensional version of this with control on the isometric distortion and the dimension of the image.

Theorem 53 (Fletcher, Kahn, Markovic)

Let $S$ be a closed Riemann surface of genus $g \geq 2$ with injectivity radius $\epsilon > 0$. Then there exists a universal constant $\alpha \geq 1$, a constant $K = K(\epsilon)$ and an $\alpha$-bi-Lipschitz homeomorphism $F : Q(S) \to V$, where $V$ is a linear subspace of $\mathbb{C}^n$ for $n \leq Kg$. 

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Infinite dimensional Teichmüller spaces
Step (iii), proving Theorem 53

- Start by covering $S$ with small balls $B_i$, for $i = 1, \ldots, n$, so that each $B_i$ is contained in a coordinate chart $\psi_i : B_i \rightarrow U_i \subset \mathbb{C}$ and further, the expression $\rho_S^{-2}\varphi$ does not vary too much when taken in the coordinate chart $U_i$.

- We can ensure that the number of such balls needed is at most $Kg$ for some $K = K(\epsilon)$. For each $i$, choose a point $p_i \in B_i$ with corresponding $z_i \in U_i$.

- Then for $\varphi \in Q(S)$, let $F(\varphi) = (f_1, \ldots, f_n) \in \mathbb{C}^n$, where $f_i = (\rho_S^{-2}\varphi)(z_i)$ in the corresponding chart $U_i$.

- Note that $F$ does depend on the choice of chart, but the norm of $F$, that is the isometric distortion, does not since $\rho_S^{-2}|\varphi|$ is a function on $S$.

- By construction, $F$ has the required properties.
Theorem 53 reduces the problem to estimating $\eta$ on $\mathbb{C}^n$ with the supremum norm, but this is straight-forward:

$$\eta(B_{\mathbb{C}^n}(0, R), r) \leq \left( \frac{2\sqrt{2}R}{r} + 2 \right)^{2n}.$$

This completes the proof of Theorem 48 and hence of Theorem 47.
The recently proved Ehrenpreis Conjecture states that given two closed Riemann surfaces $S$ and $M$ and any $K > 1$, one can find finite unbranched covers $S_1$ and $M_1$, of $S$ and $M$ respectively, such that there exists a $K$-quasiconformal map $f : S_1 \to M_1$.

One of many equivalent formulations of this result states that given any $\xi > 0$ and a closed Riemann surface $S$, there exists a finite cover $S_1$ of $S$, such that $S_1$ admits a tiling into $\xi$-nearly equilateral right angled hexagons, that is $(1 + \xi)$-quasi-isometric to the standard equilateral right angled hexagons. Given a Riemann surface $S$, the resulting cover $S'$ will typically have large injectivity radius.

One can ask if the secret as to why the Ehrenpreis conjecture holds is because once we fix $\xi > 0$ and a pattern then any closed Riemann surface $X$ of sufficiently large injectivity radius can be tiled into polygons that are $\xi$-close to the given pattern.
Question 54

Let $S_0$ denote a closed Riemann surface. Are there constants $I = I(S_0) > 0$ and $d = d(S_0) > 0$, such that every closed Riemann surface $X$, whose injectivity radius at every point is greater than $I$, is at a distance $\leq d$ from a finite cover of $S_0$?

The answer is negative.

Corollary 55

Let $S$ be a closed Riemann surface. Then there exists a universal constant $\delta_0 > 0$ such that every ball of radius 1 in every $T_g$ contains a Riemann surface that is at least $\delta_0$ away from any finite cover of $S$. 
