Gaussian integer points of analytic functions in a half-plane

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(Received )

Abstract

A classical result of Pólya states that $2^z$ is the slowest growing transcendental entire function taking integer values on the non-negative integers. Langley generalised this result to show that $2^z$ is the slowest growing transcendental function in the closed right half-plane $\Omega = \{ z \in \mathbb{C} : \Re(z) \geq 0 \}$ taking integer values on the non-negative integers. Let $E$ be a subset of the Gaussian integers in the open right half-plane with positive lower density and let $f$ be an analytic function in $\Omega$ taking values in the Gaussian integers on $E$. Then in this paper we prove that if $f$ does not grow too rapidly, then $f$ must be a polynomial. More precisely, there exists $L > 0$ such that if either the order of growth of $f$ is less than 2 or the order of growth is 2 and the type is less than $L$, then $f$ is a polynomial.

1. Introduction

Let $\Omega = \{ z \in \mathbb{C} : \Re(z) \geq 0 \}$ be the closed right half-plane and let $f$ be analytic in $\Omega$. For $r > 0$, define

$$M_\Omega(r, f) = \sup \{|f(z)| : |z| \leq r, z \in \Omega\}.$$  

Analogously to entire functions, the order $\rho$ of $f$ is

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ M_\Omega(r, f)}{\log r},$$  

where $\log^+ x = \max\{\log x, 0\}$, and the type $\lambda$ of $f$ with respect to the order is

$$\limsup_{r \to \infty} \frac{\log^+ M_\Omega(r, f)}{r^\lambda}.$$  

We say that $f$ is of exponential type if $\rho = 1$ and $\lambda < \infty$. Let $E$ be a subset of the positive integers with positive lower density $D$. Hence

$$|E \cap \{1, \ldots, n\}| \geq Dn(1 - o(1))$$

as $n \to \infty$, where $|X|$ denotes the number of elements of the set $X$. In particular, for any $D_1 < D$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|E \cap \{1, \ldots, n\}| \geq D_1 n.$$  

The following theorem was proved in [5].
Theorem 1.1. Let $E \subset \mathbb{N}$ have lower density greater than $D > 0$. Then there exists $L > 0$ depending only on $D$ such that if $f$ is analytic in $\Omega$ of exponential type $\lambda < L$ and satisfying $f(n) \in \mathbb{Z}$ for all $n \in E$, then $f$ is a polynomial.

This theorem follows in the tradition of the classical Pólya result [11] which says that $2^z$ is the slowest growing transcendental entire function which takes integer values on the non-negative integers. Further results on integer valued entire functions may be found in [3, 4, 8, 9, 10, 12, 13, 14, 15]. Recall that the Gaussian integers are $\mathbb{Z} + i\mathbb{Z} = \{m + in : m, n \in \mathbb{Z}\}$. In this article, we will prove a similar result to Theorem 1.1 for analytic functions in a half-plane which take values in the Gaussian integers at subsets of the Gaussian integers. See [15] for some results on entire functions which take integer values at the Gaussian integers.

Let $\Omega^o$ be the open right half-plane. The counting function for the Gaussian integers in $\Omega^o$ is

$$N(r) = |(\mathbb{Z} + i\mathbb{Z}) \cap \{ |z| \leq r \} \cap \Omega^o| = \frac{\pi r^2}{2} (1 + o(1))$$

as $r \to \infty$. We say that a subset $E$ of $\Omega^o \cap (\mathbb{Z} + i\mathbb{Z})$ has lower density greater than $D > 0$ if

$$\liminf_{r \to \infty} \frac{|E \cap \{ |z| \leq r \}|}{N(r)} = D.$$

Letting $N_E(r)$ be the counting function of $E$, this means that for all $D_1 < D$, there exists $r_1 > 0$ such that

$$N_E(r) \geq \frac{D_1 \pi r^2}{2}$$

for all $r \geq r_1$.

Theorem 1.2. Let $0 < D < 1$. Then there exists $L > 0$ depending only on $D$ such that the following is true. Let $E \subset \Omega^o \cap (\mathbb{Z} + i\mathbb{Z})$ have lower density greater than $D$. Let the function $f$ be analytic in the right half-plane $\Omega$, satisfy

$$\limsup_{r \to \infty} \frac{\log^+ M_{\Omega}(r, f)}{r^2} < \lambda$$

with $\lambda < L$ and assume that $f(w) \in \mathbb{Z} + i\mathbb{Z}$ for every $w \in E$. Then $f$ is a polynomial.

Remark 1.1. The proof of Theorem 1.2 shows that we may take

$$L = \frac{\pi D^2 \log(1/C_0(D))}{256}$$

where

$$C_0(D) = \frac{e^{2\psi(D)} - 1}{e^{2\psi(D)} + 1}$$

and

$$\psi(D) = \frac{8}{\pi(2 - D)} \left( \left( \frac{1}{4} \log \frac{8}{D} \right)^2 + \left( \frac{\pi^2 - 4D}{4} \right)^2 \right)^{1/2} \log \frac{4 - D}{D}. \quad (1.4)$$

We note that for $D = 1$, $L \approx 0.0001$, and that $L \to 0$ slightly faster than $D^2$ as $D \to 0$.

Remark 1.2. The Weierstrass $\sigma$-function associated to the lattice $\Lambda$ of Gaussian integers is given by

$$\sigma(z) = z \prod_{w \in \Lambda \setminus \{0\}} \left( 1 - \frac{z}{w} \right) \exp \left\{ \frac{z}{w} + \frac{z^2}{2w^2} \right\}.$$
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The function $\sigma$ restricted to $\Omega$ has order 2, bounded type (see for example [1], page 4) and zeros at all the Gaussian integer points in $\Omega$. This shows that Theorem 1·2 obtains the correct minimum order of growth for a function taking values in the Gaussian integers at all the Gaussian integers which is not a polynomial.

**Remark 1·3.** The proof of Theorem 1·2 uses the fact that $\mathbb{Z} + i\mathbb{Z}$ is closed under addition and multiplication. We therefore have an analogous result to Theorem 1·2 for analytic functions which take values in the Eisenstein integers $\mathbb{Z} + e^{2i\pi/3}\mathbb{Z}$ at subsets of the Eisenstein integers in $\Omega$, since the Eisenstein integers are also closed under addition and multiplication.

**Remark 1·4.** We need $f$ to be analytic in $\Omega$ as opposed to just $\Omega^o$ in order to apply the Nevanlinna half-plane characteristic outlined in §2.3.

**2. Lemmas needed for the proof of Theorem 1·2**

**2·1. Siegel’s linear form lemma**

The following lemma on linear forms is proved in [5, 6].

**Lemma 2·1.** Suppose $L_1, \ldots, L_m$ are linear forms with real coefficients $a_{j,k}$, for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ where $n > m$, in the $n$ variables $x_1, \ldots, x_n$. That is,

$$L_j = a_{j,1}x_1 + \ldots + a_{j,n}x_n$$

and, further, let the $a_{j,k}$ satisfy

$$\max_{j,k} |a_{j,k}| \leq A.$$

Then for $N \in \mathbb{N}$, $N > 1$, there exist integers $x_1, \ldots, x_n$ not all zero, with $|x_k| \leq X$ for $k = 1, \ldots, n$, such that

$$|L_j| \leq \frac{1}{N}$$

for $j = 1, \ldots, m$, and

$$X < 2(2nAN)^{\frac{m}{n}}.$$

We prove here a Gaussian integer analogue of this lemma. Note that a full generalization of this lemma can be found in [6].

**Lemma 2·2.** Suppose the coefficients $a_{j,k}$ of the linear forms

$$L_j = a_{j,1}x_1 + \ldots + a_{j,q}x_q,$$

for $j = 1, \ldots, p$ and where $p < q$, are integers in the ring $\mathbb{Z} + i\mathbb{Z}$ and satisfy $|a_{j,k}| < A$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$. Then there exists a solution $(x_k)_{k=1}^q$ with the $x_k$ not all zero, of the system of equations $L_j = 0$, for $j = 1, \ldots, p$, in Gaussian integers with

$$|x_k| < 2(8qA)^{p/(q-p)}. \quad (2·1)$$

**Proof.** Write

$$a_{j,k} = b_{j,k} + ic_{j,k}$$

and

$$x_k = u_k + iv_k,$$

where $b_{j,k}, c_{j,k}, u_k, v_k \in \mathbb{Z}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$. Then we have

$$L_j = \sum_{k=1}^q (b_{j,k} + ic_{j,k})(u_k + iv_k)$$
A necessary and sufficient condition for \( L_j = 0 \), \( j = 1, \ldots, p \), is that

\[
L_j,1 = \sum_{k=1}^{q} (b_{j,k} u_k - c_{j,k} v_k) = 0
\]

and

\[
L_j,2 = \sum_{k=1}^{q} (b_{j,k} v_k + c_{j,k} u_k) = 0
\]

for \( j = 1, \ldots, p \). The number of equations in this system is \( 2p \) and the number of variables is \( 2q > 2p \). Putting \( N = 2 \), \( n = 2q \), \( m = 2p \) in Lemma 2·1 yields a system of integers \( x_{r,s} \) for \( r = 1, \ldots, q \), \( s = 1, 2 \), not all of which are zero, satisfying \( x_{r,1} = u_r \), \( x_{r,2} = v_r \) and

\[
|x_{r,s}| < 2(8qA)^{p/(q-p)} \tag{2·2}
\]

so that \( |L_j,s| < 1/2 \) for \( j = 1, \ldots, p \) and \( s = 1, 2 \). However, since \( b_{j,k}, c_{j,k}, u_k, v_k \) are all integers, this means that \( L_j,s = 0 \) for \( j = 1, \ldots, p \) and \( s = 1, 2 \). This completes the proof of the lemma.

2·2. Riemann mappings

Lemma 2·3. Let \( 0 < D \leq 1 \) and define \( \theta = \pi(1-D/2) \in (0, \pi) \) and \( \beta = \sqrt{D/2} \in (0, 1) \). Let \( R > 0 \) and define

\[
V = \left\{ z \in \Omega : \frac{|z|}{2} < \beta R, |\arg z| < \frac{\pi + \theta}{4} \right\}
\]

and

\[
U = \left\{ z \in \Omega : \beta R < |z| < 2R, |\arg z| < \frac{\theta}{2} \right\}.
\]

Note that \( \overline{U} \subset V \) and \( V \subset \Omega \). Also let \( \sigma_\zeta : V \to \mathbb{D} \) be the Riemann mapping satisfying \( \sigma_\zeta(0) = 0 \) and \( \sigma_\zeta'(0) > 0 \) for some \( \zeta \in U \). Then there exists a constant \( 0 < C_0 < 1 \) depending only on \( D \) (in particular, not depending on \( R \)) such that

\[
|\sigma_\zeta(\eta)| \leq C_0 \tag{2·3}
\]

for all \( \zeta, \eta \in U \). We can take \( C_0(D) \) given by (1·3) where \( \psi(D) \) is given by (1·4).

Proof. Note first that

\[
h(z) = \log \frac{z}{R} - \frac{1}{2} \log 2\beta
\]

maps \( V \) conformally onto the rectangle

\[
h(V) = \left( -\frac{1}{2} \log \frac{8}{\beta}, 0 \right) \times \left( 0, \frac{\pi + \theta}{4}, \frac{\pi + \theta}{4} \right)
\]

and \( U \) conformally onto the rectangle

\[
h(U) = \left( -\frac{1}{2} \log \frac{2}{\beta}, 0 \right) \times \left( 0, \frac{\theta}{2}, \frac{\theta}{2} \right),
\]

so we just have to consider Riemann maps from \( h(V) \) to \( \mathbb{D} \). Let \( \tau_0 : h(V) \to \mathbb{D} \) be the Riemann map satisfying \( \tau_0(0) = 0 \) and \( \tau_0'(0) > 0 \). Writing \( d_1, d_2 \) for the hyperbolic
metrics on \( h(V) \) and \( \mathbb{D} \) respectively, where \( d_2 \) is the metric with density \( 2(1 - |z|^2)^{-1} \), we have

\[
d_2(\tau_0(z), \tau_0(0)) \leq d_1(z, 0)
\]

for any \( z \in h(V) \). Now let \( z \in \overline{h(U)} \) and \( \gamma_z \) be a path from 0 to \( z \) in \( \overline{h(U)} \) given by

\[
\gamma_z(t) = tz, \quad t \in [0, 1].
\]

Then

\[
d_1(z, 0) \leq \int_0^1 \rho_1(\gamma_z(t))|z| \, dt \quad (2.4)
\]

by the definition of hyperbolic distance, where \( \rho_1 \) is the hyperbolic density on \( h(V) \). Let \( \delta_1(w) \) be the distance from \( w \in h(V) \) to \( \partial h(V) \). Then \( 1/\delta_1 \) is the quasi-hyperbolic density on \( h(V) \), and as is well-known (see for example Theorem 8.2, [2]) \( \rho_1 \leq 2/\delta_1 \) in any simply connected region. Therefore (2.4) becomes

\[
d_1(z, 0) \leq \int_0^1 \frac{2}{\delta_1(\gamma_z(t))}|z| \, dt. \quad (2.5)
\]

It is easy to see that the right hand side of (2.5) is maximised over the closure \( \overline{h(U)} \) when \( z \) is one of the corner points of \( h(U) \). By the symmetry of the rectangle, we can consider the top right corner

\[
z_0 = \frac{1}{2} \log \frac{2}{\beta} + \frac{i\theta}{2}. \quad (2.6)
\]

For \( 0 \leq t \leq 1 \), the distance from \( \gamma_{z_0}(t) \) to the right hand edge of \( h(V) \) is

\[
\frac{1}{2} \log \frac{8}{\beta} - \frac{t}{2} \log \frac{2}{\beta}, \quad (2.7)
\]

and the distance from \( \gamma_{z_0}(t) \) to the top edge of \( h(V) \) is

\[
\frac{\pi + \theta}{4} - \frac{t\theta}{2}. \quad (2.8)
\]

We want to show that on \( \gamma_{z_0}(t) \), the shortest distance to the boundary of \( h(V) \) is always the distance to the top edge of \( h(V) \), i.e., that (2.7) is always larger than (2.8). Since \( \gamma_{z_0} \) and the edges of \( h(V) \) are linear, we just have to look at the endpoints \( \gamma_{z_0}(0) \) and \( \gamma_{z_0}(1) \) to see if this inequality holds. Putting \( \theta = \pi(1 - D/2) \) and \( \beta = \sqrt{D/2} \) in (2.7) and (2.8), for \( t = 0 \) we clearly have

\[
\log 4 > \frac{\pi D}{4}
\]

since the left hand side is greater than 1 and the right hand side is less than 1 for \( 0 < D \leq 1 \). Now, for \( t = 1 \), put

\[
\varphi(D) = \log(8\sqrt{2}) - \pi - \frac{1}{2} \log D + \frac{\pi D}{4}.
\]

We have to show that \( \varphi(D) > 0 \) for \( 0 < D \leq 1 \). Observe that

\[
\varphi(1) = \log(8\sqrt{2}) - \pi + \pi/4 \approx 0.07.
\]

Also note that \( \varphi(D) \to +\infty \) as \( D \to 0 \). Further, \( \varphi \) has one critical point \( D_0 = 2/\pi \) where \( \varphi(D_0) \approx 0.01 > 0 \). This shows that \( \varphi(D) > 0 \) for all \( 0 < D \leq 1 \) and we can conclude
that (2.7) is always larger than (2.8). Therefore,
\[
\delta_1(\gamma_w(t)) = \frac{\pi + \theta - 2t\theta}{4}.
\] (2.9)
Combining (2.5), (2.6) and (2.9) gives
\[
d_1(z,0) \leq \left(\frac{1}{2} \log \frac{2}{\beta}\right)^2 + \left(\frac{\theta}{2}\right)^2 \int_0^1 \frac{8}{\pi + \theta - 2t\theta} \, dt
= \frac{4}{\theta} \left(\frac{1}{2} \log \frac{2}{\beta}\right)^2 + \left(\frac{\theta}{2}\right)^2 \log \frac{\pi + \theta}{\pi - \theta},
\]
Putting \( \beta = \sqrt{D/2} \) and \( \theta = \pi(1-D/2) \) for \( 0 < D \leq 1 \) gives
\[
d_1(z,0) \leq \frac{8}{\pi(2-D)} \left(\frac{1}{4} \log \frac{8}{D}\right)^2 + \left(\frac{\pi(2-D)}{4}\right)^2 \log \frac{4-D}{D} = \psi(D),
\]
where \( \psi(D) \) is given in (1-4). Therefore
\[
d_2(\tau_0(z),\tau_0(0)) = d_2(\tau_0(z),0) \leq \psi(D). \quad (2.10)
\]
Now, for \( w \in h(U) \), consider the Riemann map \( \tau_w : h(V) \to D \) where \( \tau_w(w) = 0 \) and \( \tau'_w(w) > 0 \). If \( A_w \) is the Möbius transformation from \( D \) to \( D \) which maps \( \tau_0(w) \) to 0 and such that \((A \circ \tau_0)'(w) > 0\), then \( \tau_w = A_w \circ \tau_0 \). For \( z,w \in h(U) \),
\[
d_2(\tau_w(z),0) = d_2((A_w \circ \tau_0)(z),0) = d_2(\tau_0(z),A_w^{-1}(0))
= d_2(\tau_0(z),\tau_0(w)) \leq d_2(\tau_0(z),0) + d_2(\tau_0(w),0) \leq 2\psi(D), \quad (2.11)
\]
using (2.10) and the fact the Möbius transformations are hyperbolic isometries. Since
\[
d_2(\tau_w(z),0) = \log \frac{1 + |\tau_w(z)|}{1 - |\tau_w(z)|},
\]
(2.11) implies that
\[
|\tau_w(z)| \leq \frac{e^{2\psi(D)} - 1}{e^{2\psi(D)} + 1} \quad (2.12)
\]
for \( w,z \in h(U) \). Since \( \sigma_{\zeta} = \tau_{h(\zeta)} \circ h \), taking \( w = h(\zeta) \) and \( z = h(\eta) \) in (2.12) gives (2.3), which proves the lemma.

**Remark 2.1.** Note that \( \psi(1) \simeq 2.63 \) and \( \psi \to \infty \) as \( D \to 0^+ \). Similarly \( C_0(1) \simeq 0.99 \) and \( C_0 \to 1 \) as \( D \to 0^+ \). We also note that if \( \psi \) does not increase as \( D \) decreases, then we can take
\[
\psi_1(D) = \max_{t \in [D,1]} \psi(t),
\]
which does increase as \( D \) decreases. Similarly, we can assume that \( C_0 \) increases as \( D \) decreases.

2.3. The Nevanlinna characteristic in the half-plane

We will give a brief overview of the half plane characteristic analogous to the Nevanlinna characteristic in the plane. For further details, see [7]. Let \( f \) be meromorphic in \( \Omega \)
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with poles at $r_n e^{i \psi_n}$ where $r_n \geq 0$ and $-\pi/2 \leq \psi_n \leq \pi/2$. The pole counting function is

$$c(r, f) = \sum_{1 < r_n \leq r} \cos \psi_n,$$

where the sum is taken over poles of $f$ with modulus at most $r$. The integrated pole counting function is

$$C(r, f) = 2 \int_1^r c(t, f) \left( \frac{1}{t^2} + \frac{1}{r^2} \right) dt = 2 \sum_{1 < r_n \leq r} \left( \frac{1}{r_n} - \frac{r_n}{r^2} \right) \cos \psi_n.$$

The analogue to the Nevanlinna proximity function is given by the following two functions,

$$A(r, f) = \frac{1}{\pi} \int_1^r \left( \frac{1}{t^2} - \frac{1}{r^2} \right) \left[ \log^+ |f(it)| + \log^+ |f(-it)| \right] dt,$$

$$B(r, f) = \frac{2}{\pi r} \int_{-\pi/2}^{\pi/2} \log^+ |f(re^{i\phi})| \cos \phi d\phi.$$

The half-plane characteristic is then given by

$$T(r, f) = A(r, f) + B(r, f) + C(r, f).$$

The half-plane characteristic satisfies a first fundamental theorem. That is, for $a \in \mathbb{C}$ and $f$ not identically equal to $a$,

$$T \left( r, \frac{1}{f-a} \right) = T(r, f) + O(1) \quad (2.13)$$

as $r \to \infty$. For $0 < \theta < \pi$, let $S$ be the sectorial region

$$S = \{ z \in \Omega : |\arg z| < \theta/2 \}. \quad (2.14)$$

Let $E$ be a subset of the Gaussian integers in $S$ with lower density greater than $D > 0$. That is, there exists $r_1 > 0$ such that

$$N_E(r) \geq \frac{D \pi r^2}{2} \quad (2.15)$$

for all $r \geq r_1$.

**Lemma 2.4.** Let $E$ be a subset of the Gaussian integers in the sectorial region $S$, defined by (2.14), with lower density greater than $D > 0$ in $S$. Let $f$ be analytic in $\Omega$ with growth condition (1.1) and satisfying $f(z) = 0$ for all $z \in E$. If

$$\lambda < \frac{D \pi^2 \cos(\theta/2)}{4} \quad (2.16)$$

then $f \equiv 0$.

**Proof.** By the hypothesis,

$$\log^+ M_\Omega(r, f) \leq (\lambda + o(1))r^2$$

for large $r$. Therefore, for large $r$ we have

$$A(r, f) \leq \frac{1}{\pi} \int_1^r 2\lambda t^2 \left( \frac{1}{t^2} - \frac{1}{r^2} \right) dt + o(r) \leq \frac{4\lambda r}{3\pi} + o(r)$$
and
\[ B(r, f) \leq \frac{2}{\pi r} \int_{-\pi/2}^{\pi/2} \lambda r^2 \cos \phi \, d\phi + o(r) = \frac{4\lambda r}{\pi} + o(r). \]

Since \( f \) has no poles in \( \Omega \), \( C(r, f) \equiv 0 \). Therefore
\[ T(r, f) \leq \frac{16\lambda r}{3\pi} (1 + o(1)) \]
as \( r \to \infty \) and by (2.13),
\[ T(r, 1/f) \leq \frac{16\lambda r}{3\pi} (1 + o(1)) \quad (2.17) \]
as \( r \to \infty \). Now, writing the zeros of \( f \) as \( r_n e^{i\psi_n} \) where \( r_n > 0 \), \( r_n \geq r_{n-1} \) and also \(-\theta/2 \leq \psi_n \leq \theta/2,\)
\[ c(r, 1/f) = \sum_{1 < r_n \leq r} \cos \psi_n \geq \cos(\theta/2) N_E(r) \geq \frac{Dr^2 \pi \cos(\theta/2)}{2} \]
for \( r > r_1 \) by (2.15). Therefore,
\[ C(r, 1/f) \geq 2 \int_1^r \frac{Dl^2 \pi \cos(\theta/2)}{2} \left( \frac{1}{l^2} + \frac{1}{r^2} \right) \, dl - O(1) \]
\[ = \frac{4Dl^\pi \cos(\theta/2)}{3} (1 - o(1)) \quad (2.18) \]
as \( r \to \infty \). Since \( C(r, 1/f) \leq T(r, 1/f) \), then combining (2.17) and (2.18) gives
\[ \lambda \geq \frac{Dl^\pi \cos(\theta/2)}{4}. \]
This contradicts (2.16) which implies that \( f \) must be identically zero and proves the lemma.

2.4. Algebraic functions mapping Gaussian integers to Gaussian integers

For \( 0 < \theta < \pi \), let \( S \) be the sectorial region \( \{z \in \Omega : |\arg z| < \theta/2\} \) and let \( E \) be a subset of the Gaussian integers in \( S \) with lower density greater than \( D > 0 \) in \( \Omega \), in particular satisfying (2.15). Let \( f \) be analytic and algebraic such that \( f(w) \in \mathbb{Z} + i\mathbb{Z} \) for every \( w \in E \). The aim of this section is to show that \( f \) must be a polynomial. Since \( f \) is algebraic, it satisfies an equation of the form
\[ c_p(z)f(z)^p + \ldots + c_1(z)f(z) + c_0(z) \equiv 0 \quad (2.19) \]
where the \( c_j(z) \) are polynomials for \( j = 0, \ldots, p \), and \( c_0(z)c_p(z) \) is not identically zero. Let \( d_j \) be the degree of the polynomial \( c_j \) and let
\[ K = \max_{j=0,\ldots,p} d_j. \quad (2.20) \]
Since \( f \) is algebraic,
\[ M_{\Omega}(r, f) \leq O(r^K) \quad (2.21) \]
as \( r \to \infty \). To see this, write (2.19) as
\[ f^p = \frac{-c_{p-1}f^{p-1} - \ldots - c_0}{c_p}, \]
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and so

\[ M_\Omega(r, f)^p \leq C_p r^K M_\Omega(r, f)^{p-1} \]

for some constant \( C \), which gives

\[ M_\Omega(r, f) \leq O(r^K). \]

as \( r \to \infty \). Now, let \( N \in \mathbb{N} \) satisfy

\[ N > (p+1)K \tag{2.22} \]

with \( K \) and \( p \) defined as above. Further, let \( \tilde{N} \in \mathbb{N} \) satisfy

\[ \frac{D\pi\tilde{N}^2}{\theta} \geq N + 1. \tag{2.23} \]

**Lemma 2.5.** There exist \( w \in (\mathbb{Z} + i\mathbb{Z}) \cap S \) with \(|w|\) arbitrarily large such that

\[ \left| E \cap \left( [Re(w), Re(w) + \tilde{N}] \times [Im(w), Im(w) + \tilde{N}] \right) \right| \geq N \]

Proof. If this is not the case, then there exists \( r_0 \) such that for all \( w \in (\mathbb{Z} + i\mathbb{Z}) \cap S \) with \(|w| > r_0\), we have

\[ \left| E \cap \left( [Re(w), Re(w) + \tilde{N}] \times [Im(w), Im(w) + \tilde{N}] \right) \right| \leq N. \]

The area of the box \([Re(w), Re(w) + \tilde{N}] \times [Im(w), Im(w) + \tilde{N}]\) is \( \tilde{N}^2 \), and so \( S \cap \{|z| \leq r\} \) can be covered by

\[ \frac{\theta r^2(1 + o(1))}{2(\tilde{N})^2} \]

boxes of this form for large \( r \), and each box has at most \( N \) elements of \( E \) for \(|w| \geq r_0\). Using this, and since the lower density of \( E \) exceeds \( D \), we have, for large \( r \),

\[ \frac{D\pi\tilde{N}^2r^2}{2} \leq N_E(\tilde{N}r) = (N_E(\tilde{N}r) - N_E(\tilde{N}r_0)) + N_E(\tilde{N}r_0) \]

\[ \leq \frac{\theta N(\tilde{N}^2r^2 - r_0^2)}{2N^2} + O(1) = \frac{\theta NR^2(1 + o(1))}{2}. \]

This implies that

\[ \frac{D\pi\tilde{N}^2}{\theta} \leq N, \]

which contradicts (2.23) and proves the lemma.

Pick \( w \) satisfying the conclusion of Lemma 2.5 and let \( a_0, \ldots, a_N \) be elements of \( E \) in \([Re(w), Re(w) + N] \times [Im(w), Im(w) + \tilde{N}]\). Let \( \epsilon > 0 \) be small and let \( \Gamma \) be the circle with centre \( w \) and radius \( \epsilon|w| \) described once counter-clockwise. We can choose \( w \) with \(|w|\) large enough such that \( \Gamma \) is contained in \( \Omega \), \( a_0, \ldots, a_N \) lie inside \( \Gamma \) and satisfy

\[ |t - a_k| \geq \frac{\epsilon|w|}{2} \tag{2.24} \]

for \( t \in \Gamma \), \( k = 0, \ldots, N \). For \( k = 0, \ldots, N \), it follows from Cauchy’s integral formula and the identity

\[ \frac{1}{t - z} = \frac{1}{t - a_0} + \frac{z - a_0}{(t - a_0)(t - a_1)} + \ldots + \frac{(z - a_0)(z - a_k)}{(t - a_0)(t - a_1)(t - a_k)(t - z)}, \]
Lemma 2

less than 1

Assume that

where

Then

which can be proved by induction, that

for \( z \) inside \( \Gamma \), where

is given by

Thus \( P_k(z) \) is the interpolating polynomial of degree at most \( k \) which equals \( f(z) \) at the \( k + 1 \) points \( a_0, ..., a_k \). Now, let

Then \( Q \in \mathbb{Z} + i\mathbb{Z} \) and

where \( C \) is a constant independent of \( |w| \). Since \( f(a_k) \in \mathbb{Z} + i\mathbb{Z} \) for \( k = 0, ..., N \), it follows from (2.27), (2.28) and the residue theorem that \( QA_j \in \mathbb{Z} + i\mathbb{Z} \) for \( j = 0, ..., N \). Now, combining (2.21), (2.24), (2.27) and (2.28) with the fact that \( |w| \) is large,

where \( C' \) is a constant which does not depend on \( |w| \). We can therefore choose \( w \) satisfying the conclusion of Lemma 2.5 with \( |w| \) large enough to make the right hand side of (2.29) less than \( 1/2 \) for \( j > K \). However, since \( QA_j \in \mathbb{Z} + i\mathbb{Z} \), it follows that \( A_j = 0 \) for \( j > K \).

Lemma 2.6. We have \( f = P_K \) and thus \( f \) is a polynomial.

Proof. Recalling (2.25) and the definition (2.26) of \( P_k \), it now follows that \( P_K = P_N \), and \( f - P_K \) has \( N + 1 \) zeros \( a_0, ..., a_N \) in the box

Assume that \( f - P_K \) is not identically 0. Using (2.19), write

Using (2.20) and (2.22). Now \( f - P_K \) has at least \( N + 1 \) zeros and (2.30) implies that \( b_0 \) must also have at least \( N + 1 \) zeros. However, the degree of the polynomial \( b_0 \) is less than \( N \) by (2.31) and therefore \( b_0 \equiv 0 \). Since \( f - P_K \) is not identically zero, we can cancel an \( f - P_K \) from (2.30) to get

\[ c_p(f - P_K)^{p-1} + ... + b_1 = 0. \]
Note that if \( b_1 \) is identically zero, we can cancel another power of \( f - P_K \), but since \( c_n \) is not identically zero, we must eventually terminate the cancelling process. As above, \( f - P_K \) has at least \( N + 1 \) zeros, which means \( b_1 \) has at least \( N + 1 \) zeros. The degree of \( b_1 \) is less than \( N \) and so \( b_1 \) is identically zero. We continue this process to show that \( b_j \equiv 0 \) for \( j = 0, \ldots, p - 1 \). This leaves

\[
c_p(f - P_K)^q = 0
\]

for some integer \( q \geq 1 \). However, \( c_p \) is not identically zero and \( f - P_K \) is not identically zero by assumption. We therefore have a contradiction and can conclude that \( f = P_K \).

**Remark 2.2.** We will only need this lemma in the case that \( p = 1 \), i.e., when \( f \) is rational.

3. Proof of Theorem 1.2

Let \( D, \lambda, f \) be as in the hypotheses, so that for large \( r \)

\[
\log^+ M_\beta(r, f) \leq \lambda r^2. \tag{3.1}
\]

Let \( E \subset \Omega \cap (\mathbb{Z} + i\mathbb{Z}) \) have lower density greater than \( D > 0 \). Then there exists \( r_1 > 0 \) such that the counting function \( N_E(r) \) of \( E \) satisfies

\[
N_E(r) \geq \frac{D\pi r^2}{2}
\]

for all \( r \geq r_1 \). We can find \( \theta \in (0, \pi) \) such that the intersection of \( E \) with the sector

\[
S = \{ z \in \Omega : |\arg z| < \theta/2 \}
\]

has positive lower density. To see this, let

\[
\theta = \pi(1 - D/2) \tag{3.3}
\]

and let \( \tilde{N}_E(r) \) be the counting function of \( \tilde{E} = E \cap S \). The counting function \( \tilde{N}_E(r) \) for the Gaussian integers in \( \Omega \setminus S \) satisfies

\[
\tilde{N}_E(r) = \frac{(\pi - \theta)r^2(1 + o(1))}{2}
\]

for large \( r \). Then for \( r > r_1 \),

\[
N_{\tilde{E}}(r) \geq N_E(r) - \tilde{N}(r) \geq \frac{D\pi r^2}{2} - \frac{(\pi - \theta)r^2(1 + o(1))}{2} \geq \frac{D\pi r^2(1 + o(1))}{4} \tag{3.4}
\]

using (3.3). Therefore \( \tilde{E} \) has lower density at least \( D/2 \) in \( \Omega \) and is supported in \( S \) defined by (3.2), and for the rest of the proof we will work with \( \tilde{E} \) instead of \( E \). Enumerate \( \tilde{E} \) by \( \alpha_0, \alpha_1, \ldots \) with \( |\alpha_i| \leq |\alpha_{i+1}| \) and if we have equality, then \( \arg \alpha_i + \pi/2 \leq \arg \alpha_{i+1} + \pi/2 \). Choose \( R > r_1 \) to be large and set

\[
H = \frac{n}{2} = N_{\tilde{E}}(R) \in \mathbb{N}.
\]

Following the method of Waldschmidt in [13], form the functions

\[
g_{\mu, \nu}(z) = z^\mu f(z)^\nu
\]

for \( \mu = 0, 1, \ldots, H - 1 \) and \( \nu = 0, 1 \). This gives \( 2H = n \) functions which we label as \( g_1, \ldots, g_n \), where

\[
g_k(z) = z^{\mu(k)} f(z)^{\nu(k)}. \tag{3.5}
\]
The aim of the proof is to show that the functions $g_1, \ldots, g_k$ are linearly dependent over $\mathbb{Z} + i\mathbb{Z}$. This then implies that $f$ is rational, and in particular algebraic, so that Lemma 2.6 shows that $f$ is a polynomial. In order to prove that the $g_k$ are linearly dependent, observe first that

$$a_{j,k} = g_k(\alpha_j) \in \mathbb{Z} + i\mathbb{Z}$$

for $k = 1, \ldots, n$ and $j = 0, \ldots, H - 1$ since $f(w) \in \mathbb{Z} + i\mathbb{Z}$ for $w \in \tilde{E}$ and also $w^p \in \mathbb{Z} + i\mathbb{Z}$ for any Gaussian integer $w$ and positive integer $p$. Further,

$$|a_{j,k}| = |(\alpha_j)^{\mu(k)} f(\alpha_j)^{\nu(k)}| \leq R^{H-1} M_\Omega(R, f) \leq R^{H-1} e^{\lambda R^2}$$

for $k = 1, \ldots, n$ and $j = 0, \ldots, H - 1$, using the fact that $R$ is large. Applying Lemma 2.2 with $p = H$, $q = n$ and $A = [R^{H-1} e^{\lambda R^2}] + 1$ yields Gaussian integers $A_1, \ldots, A_n$ such that

$$\sum_{k=1}^n A_k g_k(\alpha_j) = 0$$

for $j = 0, \ldots, H - 1$ and

$$|A_k| \leq 16n \left( [R^{H-1} e^{\lambda R^2}] + 1 \right) \leq 32n R^{H-1} e^{\lambda R^2},$$

for $k = 1, \ldots, n$. Set

$$F(z) = \sum_{k=1}^n A_k g_k(z).$$

Clearly $F$ has zeros at $\alpha_0, \ldots, \alpha_{H-1}$. We now estimate the growth of $F$. For $r > R$, we have

$$M_\Omega(r, F) \leq 32n^2 R^{H-1} e^{\lambda R^2} r^{H-1} e^{\lambda r^2}$$

$$\leq C r^{2(H+1)} e^{2\lambda r^2}$$

where $C$ denotes a constant independent of $n$, $R$ and $r$. Here we are using (3-1) and the fact that $n = O(R^2) \leq O(r^2)$. Then

$$\log^+ M_\Omega(r, F) \leq 2(H + 1) \log r + 2\lambda r^2 + O(1) = 2\lambda r^2(1 + o(1)) \quad (3-6)$$

as $r \to \infty$. Now let

$$\beta = \sqrt{D/2}, \quad (3-7)$$

and define

$$U = \left\{ z \in \Omega : \beta R < |z| < 2R, |\arg z| < \frac{\theta}{2} \right\}$$

and

$$V = \left\{ z \in \Omega : \beta R/2 < |z| < 4R, |\arg z| < \frac{\pi + \theta}{4} \right\}.$$

Note that we can choose $R$ to be transcendental irrational so that no Gaussian integers lie on $\{z : |z| = 2^i R\}$ for $i \in \mathbb{Z}^+$.

**Lemma 3.1.** Let $z \in \tilde{E} \cap U$. Then $F(z) = 0$.

**Proof.** Now $F$ has at least $N_\tilde{E}(R) - N_S(\beta R)$ zeros in $U$ at points of $\tilde{E}$, where $N_S$ denotes the counting function of the Gaussian integers in the sector $S$ and is given by

$$N_S(r) = \frac{\theta r^2(1 + o(1))}{2}. \quad (3-8)$$
Label the zeros of $F$ in $U$ by $x_1, \ldots, x_q$, where

$$q \geq N_E(R) - N_S(\beta R) \geq \frac{D\pi R^2}{4} - \frac{\theta \beta^2 R^2(1 + o(1))}{2} = \frac{\pi D^2 R^2(1 + o(1))}{8},$$

(3-9)

using (3-4), (3-7) and (3-8). For $j = 1, \ldots, q$, let $\sigma_j : V \to \mathbb{D}$ be the Riemann map satisfying $\sigma_j(x_j) = 0$, $\sigma_j'(x_j) > 0$. Then by applying Lemma 2.3, there exists a constant $C_0 \in (0, 1)$ depending only on $D$ (recall $\theta = \pi(1 - D/2)$ and $\beta = \sqrt{D/2}$) such that

$$|\sigma_j(z)| \leq C_0$$

(3-10)

for $j = 1, \ldots, q$ and for all $z \in U$. Write

$$G(z) = F(z) \prod_{j=1}^q (\sigma_j(z))^{-1}.$$  

Then $G$ is analytic in $V$ and by the maximum principle,

$$\sup_{z \in V} |G(z)| = \sup_{z \in \partial V} |F(z)| \leq M_{\Omega}(4R, F).$$

(3-11)

Now for $z \in U$, 

$$|F(z)| \leq \sup_{w \in V} |G(w)| \prod_{j=1}^q |\sigma_j(z)|$$

$$\leq \exp\left\{32\lambda R^2(1 + o(1))\right\} C_0^q \leq \exp\left\{\left(32\lambda + \frac{\pi D^2 \log C_0}{8}\right) R^2(1 + o(1))\right\}$$

(3-12)

for large $R$, using (3-6), (3-9), (3-10) and (3-11). Now if we choose $\lambda$ small enough so that

$$\lambda < \frac{\pi D^2 \log(1/C_0)}{256},$$

(3-13)

then the coefficient of $R^2$ in (3-12) is negative (note that $\log C_0 < 0$) and by choosing $R$ large enough initially, we can make the right hand side of (3-12) less than 1. Then if $\zeta \in \tilde{E} \cap U$, we have $F(\zeta) \in \mathbb{Z} + i\mathbb{Z}$ and therefore $F(\zeta) = 0$.

We now want to continue this process of accumulating zeros of $F$. Note that the set $\tilde{E} \cap \{|z| > \beta R\}$ is obtained by deleting finitely many points from $\tilde{E}$. Therefore the lower density of $\tilde{E} \cap \{|z| > \beta R\}$ is the same as that of $\tilde{E}$.

**Lemma 3.2.** If $z \in \tilde{E} \cap \{|z| > \beta R\}$, then $F(z) = 0$.

**Proof.** Define

$$U_i = \left\{z \in \Omega : 2^i \beta R < |z| < 2^{i+1} R, |\arg z| < \frac{\theta}{2}\right\}$$

and

$$V_i = \left\{z \in \Omega : 2^i \beta R/2 < |z| < 2^{i+1} R, |\arg z| < \frac{\pi + \theta}{4}\right\}.$$  

The proof runs by induction. That is, we want to show that $F(z) = 0$ for all $z \in \tilde{E} \cap U_{i-1}$ implies that $F(z) = 0$ for all $z \in \tilde{E} \cap U_i$. Lemma 3.1 deals with the first step $i = 0$, so for the inductive step we can assume that $F$ has zeros for all $z \in \tilde{E} \cap U_{i-1}$, where $i \geq 1$.

Now the number of zeros of $F$ arising from points of $\tilde{E}$ in $\{|z| < 2^i/\beta R\}$ is at most

$$N_S(2^i \beta R) = 2^{2^i-1} \theta \beta^2 R^2(1 + o(1))$$

(3-14)
for large \( R \). Therefore, if we label the zeros of \( F \) at points of \( \tilde{E} \) in \( U_i \) inherited from \( U_{i-1} \) by \( x_1, \ldots, x_q \), then since \( R \) was chosen to be large

\[
q_i \geq N(2^{i-1} \cdot 2R) - N(2^{i+3}R) \tag{3.15}
\]

\[
\geq D\pi 2^{2i-2}R^2 - 2^{2i-1}\beta^2 R^2(1 + o(1)) = 2^{2i-3}\pi D^2 R^2(1 + o(1))
\]

by (3.4), (3.14) and using the definitions of \( \theta \) and \( \beta \). As in the proof of Lemma 3.1, for \( j = 1, \ldots, q_i \), let \( \sigma_j : V_i \to \mathbb{D} \) be the Riemann mapping satisfying \( \sigma_j(x_j) = 0 \) and \( \sigma_j'(x_j) > 0 \). Then, by applying Lemma 2.3, replacing \( R \) with \( 2^i R \), we get the same constant \( C_0 < 1 \) from the proof of Lemma 3.1 such that

\[
|\sigma_j(z)| \leq C_0 \tag{3.16}
\]

for \( j = 1, \ldots, q_i \) and for all \( z \in U_i \). Write

\[
G_i(z) = F(z) \prod_{j=1}^{q_i}(\sigma_j(z))^{-1}.
\]

Then \( G_i \) is analytic in \( V_i \) and by the maximum principle,

\[
\sup_{z \in V_i} |G_i(z)| = \sup_{z \in \partial V_i} |F(z)| \leq M_{G_1}(2^{i+2}R, F). \tag{3.17}
\]

Now, for \( z \in U_i \),

\[
|F(z)| \leq \sup_{w \in V_i} |G_i(w)| \prod_{j=1}^{q_i} |\sigma_j(z)|
\]

\[
\leq \exp \left\{ \lambda 2^{i+5}R^2(1 + o(1)) \right\} C_0^{q_i} \leq \exp \left\{ (256\lambda + \pi D^2 \log C_0)2^{2i-3}R^2(1 + o(1)) \right\} \tag{3.18}
\]

for large \( R \), using (3.6), (3.15), (3.16) and (3.17). As before, if we choose \( \lambda \) small enough to satisfy (3.13) then the coefficient of \( R^2 \) in (3.12) is negative. Since the right hand side of (3.12) is less than one, then the right hand side of (3.18) is also less than one (this is clearly true for \( i \geq 2 \), for \( i = 1 \), doubling \( R \) will allow the right hand side to be less than one without changing the bound for \( \lambda \) in (3.13)).

For \( \zeta \in \tilde{E} \cap U_i \) we have \( F(\zeta) \in \mathbb{Z} + i\mathbb{Z} \), but also \( |F(\zeta)| < 1 \) which implies that \( F(\zeta) = 0 \). This completes the inductive step and the proof of the lemma.

If we now choose

\[
L = \min \left\{ \frac{\pi D^2 \log(1/C_0)}{256}, \frac{\pi D^2 \cos(\theta/2)}{4} \right\}, \tag{3.19}
\]

then \( L \) depends only on \( D \), since \( C_0 \) and \( \theta \) depend only on \( D \). However, since \( \theta = \pi(1 - D/2) \), we have \( \cos(\theta/2) = \sin \pi D/4 \). Using this, and since \( \log 1/C_0(D) \leq \log 1/C_0(1) \approx 0.01 \) for \( 0 < D < 1 \) by Remark 2.1, we clearly have

\[
\left( \frac{\pi D^2 \log(1/C_0)}{256} \right) \left( \frac{\pi D^2 \cos(\theta/2)}{4} \right)^{-1} \leq \frac{D}{6400\pi \sin(\pi D/4)} < 1
\]

since \( D/\sin(\pi D/4) < 6400\pi \) for \( 0 < D < 1 \). Using (3.19), this gives (1.2). If \( f \) is analytic satisfying growth condition (1.1) with \( \lambda < L \) and also satisfies the hypotheses of the theorem, then by Lemma 3.2, \( F(z) = 0 \) for all \( z \in \tilde{E} \cap \{|z| > \beta R\} \). Then Lemma 2.4
Gaussian integer points of analytic functions in a half-plane implies that $F \equiv 0$. This means that

$$F(z) = \sum_{k=1}^{n} A_k g_k(z) \equiv 0$$

and therefore $f$ is rational, recalling (3·5). Finally, Lemma 2.5 implies that $f$ is a polynomial which completes the proof.

Acknowledgements. The author wishes to thank Professor Jim Langley and the referee for many extremely helpful comments and suggestions. The author was supported by EPSRC grant RA22AP.

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