1 Introduction

1.1 An Overview

A Bloch function is an analytic map from $\mathbb{D}$ to $\mathbb{C}$, which reduces lengths by a fixed factor from the hyperbolic metric on $\mathbb{D}$ to the Euclidean metric on $\mathbb{C}$. Equipped with the Bloch norm, the space of Bloch functions is a Banach space, so Bloch functions provide a link between complex analysis and function theory, highlighted in Chapter 5 by the link between the existence of inner functions in the little Bloch space and the compactness of their associated composition operators in the little Bloch space.

Chapter 1 deals with some preliminary complex analysis and introduces Bloch functions and some of their basic properties.

Chapter 2 looks at the properties of $\mathcal{B}$ and $\mathcal{B}_0$ as Banach spaces. Theorem 2.2 gives a characterisation of $\mathcal{B}_0$ as a subspace of $\mathcal{B}$. Then the dual of $\mathcal{B}_0$ is investigated and it is concluded that $\mathcal{B}_0^{**}$ can be identified with $\mathcal{B}$.

Chapter 3 looks more at some properties of Bloch functions themselves, commencing with a maximum and minimum principle for Bloch functions, Theorem 3.1, and proceeding to give some corollaries to this, which exhibit some of the boundary behaviour features of Bloch functions. The integrals of Bloch functions are then investigated, which leads to another characterisation of Bloch functions of positive real part in terms of Zygmund measures, Proposition 3.8. Finally, some results on the growth of Bloch functions are exhibited, including the Makarov law of the iterated logarithm which provides a link to probability theory.

Chapter 4 moves on to look at composition operators on the Bloch space. Section 4.1 gives some necessary and sufficient conditions for a composition operator to have closed range, or equivalently, be bounded below. Section 4.2 turns to necessary and sufficient conditions for a composition operator to be compact on $\mathcal{B}$ and $\mathcal{B}_0$ (Theorems 4.7 and 4.8 respectively). The chapter finishes with some examples showing how the boundary behaviour of a map from $\mathbb{D}$ to itself affects whether or not the associated composition operator is compact.

Chapter 5 introduces Blaschke products and singular functions as inner functions. The question of whether there are inner functions in the spaces $\mathcal{B}$, $\mathcal{B}_0$ and
\(B^0\) is addressed and the existence of compact composition operators associated to inner functions on \(B\) and \(B^0\) is established.

Chapters 1, 3 are based on [1]. Chapter 2 is based on [2]. Chapter 4 Section 1 is based on [3] and Sections 2 and 3 are based on [4]. The Hardy space and Bergman space example comes from [8]. Chapter 5 Section 1 is based on [5], and Sections 2 and 3 are based on [9] and [6].

1.2 Preliminaries

The only conformal maps of \(C^\infty\) to itself are of the form

\[
\tau(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.
\]

These are the Möbius transformations and form a group, Möb, with respect to composition. The only conformal maps of \(D = \{z \in \mathbb{C} : |z| < 1\}\) to itself are Möbius transformations of the form

\[
\tau(z) = \frac{cz + z_0}{\bar{c}z_0 + 1}, \quad |z_0| < 1, |c| = 1
\]

These form a subgroup Möb(D) = \{\(\tau \in \text{Möb} : \tau(D) = D\}\).

The Schwarz-Pick Lemma states that for \(f\) an analytic function in \(D\),

\[
|f'(z_0)| \leq \frac{1 - |f(z_0)|^2}{1 - |z_0|^2}
\]

with equality \(\iff f \in \text{Möb}(D)\). Hence if \(h = f \circ \tau\), \(f\) analytic function in \(D\) and \(\tau \in \text{Möb}(D)\), then \(h(0) = f(z_0)\) and

\[
(1 - |z|^2)|h'(z)| = (1 - |\tau(z)|^2)|f'(\tau(z))|
\]

for \(z \in D\).

The hyperbolic metric on \(D\) is given by

\[
\rho(z_1, z_2) = \min_C \int_C \frac{2|dz|}{1 - |z|^2}
\]

for \(z_1, z_2 \in D\) with the minimum being taken over all curves in \(D\) from \(z_1\) to \(z_2\) and the minimum is attained by the arc of the circle passing through \(z_1, z_2\) orthogonal to \(T\). Thus

\[
\rho(z_1, z_2) = \log \left(1 + \frac{|z_2 - z_1|}{|1 - \bar{z}_1 z_2|}\right).
\]

\(\rho\) is invariant under Möbius transformations ie.

\[
\rho(\tau(z_1), \tau(z_2)) = \rho(z_1, z_2), \quad \forall \tau \in \text{Möb}(D) \quad \text{and} \quad \forall z_1, z_2 \in D.
\]
A Stolz angle at \( \xi \in \mathbb{T} \) is of the form
\[
\Delta = \{ z \in \mathbb{D} : |\arg(1 - \xi z)| < \alpha, |z - \xi| < \beta \}
\]
for \( 0 < \alpha < \frac{\pi}{2}, \beta < 2 \cos \alpha \). The important geometrical point is that all points of \( \Delta \) have bounded hyperbolic distance from the radial line segment \([0, \xi]\).

If \( f : \mathbb{D} \to \mathbb{C}_\infty \),

(i) \( f \) has angular limit \( a \in \mathbb{C}_\infty \) at \( \xi \in \mathbb{T} \) if \( f(z) \to a \) as \( z \to \xi, z \in \Delta \), for each Stolz angle \( \Delta \) at \( \xi \). \( f(\xi) \) denotes an angular limit where it exists.

(ii) \( f \) has unrestricted limit \( a \in \mathbb{C}_\infty \) at \( \xi \in \mathbb{T} \) if \( f(z) \to a \) as \( z \to \xi, z \in \mathbb{D} \).

Then define \( f(\xi) = a \) and \( f \) is continuous at \( \xi \) as a function on \( \mathbb{D} \cup \{ \xi \} \).

(iii) \( f \) has asymptotic value \( a \in \mathbb{C}_\infty \) at \( \xi \in \mathbb{T} \) if \( f(z) \to a \) as \( z \to \xi, z \in \Gamma \) for some curve \( \Gamma \) contained in \( \mathbb{D} \).

EXAMPLE 1) An angular limit does not imply the existence of an unrestricted limit.

For \( z \in \mathbb{D} \), \( f_0 = \exp(-\frac{1+z}{1-z}) \) has angular limit 0 at 1. But \( \text{Re}(\frac{1+z}{1-z}) = \frac{1-z^2}{1-z^2} \) for \( z = x + iy \). So if \( z \to 1 \) along the circle tangential to \( \mathbb{T} \) given by \( (x - \frac{\alpha}{\alpha+1})^2 + y^2 = (\alpha + 1)^2 \), then \( \text{Re}(\frac{1+z}{1-z}) = \alpha \), i.e. a constant, so also \( |f_0(z)| \) is constant. Thus \( f_0 \) cannot have an unrestricted limit.

Let \( S \) be the space \( \{ f(z) = z + a_2z^2 + \ldots : f \text{ analytic and univalent in } \mathbb{D} \} \).

Bieberbach \( \Rightarrow |a_n| \leq n, \forall n \).

The Koebe transform,
\[
h(z) = \frac{f(z + \xi z_0) - f(z_0)}{(1 - |z_0|^2)f'(z_0)} = z + \left( \frac{1}{2} (1 - |z_0|^2) \frac{f''(z_0)}{f'(z_0)} - \frac{z_0}{2} \right) z^2 + \ldots
\]
allows information at 0 to be transferred to information at any \( z_0 \in \mathbb{D} \).

\[
\implies \left| (1 - |z|^2) \frac{f''(z)}{f'(z)} - 2\pi \right| \leq 4, \forall z \in \mathbb{D}
\]

(1)

Integrating this inequality shows that
\[
|f'(0)| \frac{|z|}{1 + |z|^2} \leq |f(z) - f(0)| \leq |f'(0)| \frac{|z|}{1 - |z|^2}.
\]

(2)

For \( h \) as above and from (2),
\[
\frac{1}{4} \leq \liminf_{|z| \to 1} \left| \frac{h(z)}{z} \right| \leq |h'(0)| = 1.
\]

Define for \( z \in \mathbb{D} \), \( d_f(z) = \text{dist}(f(z), \partial f(\mathbb{D})) \). Then
\[
d_f(z_0) = \liminf_{|\xi| \to 1} |f(\xi) - f(z_0)|
\]

3
Proof. Suppose \( f \) is a Bloch function if it is analytic in \( \mathbb{D} \) and if

\[
\|f\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.
\]

There are several other characterisations of Bloch functions:

(i) \( \sup_{z \in \mathbb{D}} df(z) < \infty \)

This follows from (3)

(ii) \( \{ f_\tau(z) = f(\tau(z)) : \tau \in \text{Möb}(\mathbb{D}) \} \) is finitely normal in \( \mathbb{D} \).

Since \((1 - |z|^2)|f_\tau'(z)| = (1 - |\tau(z)|^2)|f'(\tau(z))| \) \( \forall f_\tau \), it follows that

\[ \sup_{z \in \mathbb{D}} (1 - |z|^2)|f_\tau'(z)| < \infty. \]

(iii) \( \exists \) \( g \) analytic and univalent in \( \mathbb{D} \) and a constant \( \alpha > 0 \) such that

\[ f(z) = \alpha \log g'(z). \]  

(4)

Proof. Suppose \( f(z) = \alpha \log g'(z) \), then \((1 - |z|^2)|f'(z)| = \alpha(1 - |z|^2) \left| \frac{g''(z)}{g'(z)} \right| \).

If \( g(z) \) univalent in \( \mathbb{D} \), \( \left| \frac{g''(z)}{g'(z)} \right| \leq \frac{6}{(1 - |z|^2)} \).

So \( \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| \leq 6\alpha \).

Conversely, define \( \frac{1}{\alpha} = 3 \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|, \) \( g(z) = \int_0^z \exp \left[ \frac{f(\xi)}{\alpha} \right] d\xi \).

Then (4) gives \((1 - |z|^2) \left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{2} \) \( \forall z \in \mathbb{D} \).

Becker Univalence Criterion \( \Rightarrow \) \( g(z) \) univalent in \( \mathbb{D} \).

Denote by \( \mathcal{B} \) the space of Bloch functions and by \( \mathcal{B}_0 \), the space of Bloch functions satisfying

\[ \lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0. \]

In section 2, the Banach space properties of \( \mathcal{B} \) will be investigated. For now, note that \( \| \cdot \|_B \) is a semi-norm, i.e. satisfies all the conditions for a norm except that \( \|f\|_B = 0 \) does not imply \( f \equiv 0 \) eg. take a constant function.

From Schwarz-Pick,

\[ \left| (1 - |z|^2) \frac{d}{dz} f(\tau(z)) \right| = (1 - |\tau(z)|^2)|f'(\tau(z))| \quad \forall z \in \mathbb{D}, \tau \in \text{Möb}(\mathbb{D}) \]
The left hand side is $||f \circ \tau||_B$ when taking the supremum, and similarly, the right hand side is $||f||_B$. Thus $||f \circ \tau||_B = ||f||_B$ $\tau \in \text{Möb}(\mathbb{D})$. i.e $||.||_B$ is conformally invariant.

Proceeding $f$ by an analytic function $g : \mathbb{C} \to \mathbb{C}$ gives a Bloch function if and only if sup$_{z \in D} |g'(f(z))| < \infty$. So, for example, if $g$ is a Euclidean similarity of $\mathbb{C}$ or a scaling of $\mathbb{C}$ then $g \circ f$ is a Bloch function.

Bloch functions satisfy a useful inequality relating the hyperbolic distance of two points and the euclidean distance of their images. For $f \in B$ and $z \in \mathbb{D}$,

$$|f(z) - f(0)| = \left| z \int_0^1 f'(rz)dr \right| \leq \left| z \int_0^1 \frac{||f||_B}{1 - r^2z^2}dr \right| \leq ||f||_B \int_0^1 \frac{|z|}{1 - r^2|z|^2}dr$$

$$\Rightarrow \left| f(z) - f(0) \right| \leq \frac{1}{2}||f||_B \log \left( \frac{1 + |z|}{1 - |z|} \right) = ||f||_B \rho(z,0)$$

where $\rho$ is the hyperbolic metric. But since $||f \circ \tau||_B = ||f||_B$, take $\tau(z) = \frac{z + z_1}{1 + z_1 z_2}$ to give

$$|f(z_1) - f(z_2)| \leq ||f||_B \rho(z_1, z_2) \quad \forall z_1, z_2 \in \mathbb{D} \quad (5)$$

Thus Bloch functions reduce lengths by a fixed factor from the hyperbolic metric in $D$ to the Euclidean metric in $\mathbb{C}$.

**EXAMPLE 2** Examples of Bloch functions.

Schwarz-Pick $\Rightarrow (1 - |z|^2)|g'(z)| \leq 1 - |g(z)|^2$. So if $g$ is bounded and analytic in $D$, then $g \in B$. Moreover, if $|g(z)| \leq 1 \forall z \in \mathbb{D}$, then $||g||_B \leq 1$.

However, there do exist unbounded Bloch functions. Take $h(z) = \log(\frac{1 + z}{1 - z})$.

Then $h'(z) = \frac{2}{1 - z^2} \Rightarrow ||h||_B = 2$ and so $h \in B$.

Note also $|h(z) - h(0)| = ||h||_B \rho(z,0)$, i.e equality in (5) is attained. However, $h$ is unbounded on $\mathbb{D}$.

**Proposition 1.1.** If $f : D \to \mathbb{C}$ is conformal then

(i) $||\log(f')||_B \leq 6$.

(ii) $||\log(f - a)||_B \leq 4$ for $a \notin f(D)$.

Proof. (1) $\Rightarrow (1 - |z|^2)\frac{d}{dz} \log(f'(z))) = (1 - |z|^2)\frac{f''(z)}{f'(z)} \leq 4 + 2|z| < 6$.

So $||\log(f')||_B \leq 6$, to give (i).

For (ii), note $|f(z) - a| \geq df(z)$ for $a \notin f(D)$.

Then (3) $\Rightarrow (1 - |z|^2)\frac{d}{dz} \log(f(z) - a) = \frac{(1-|z|^2)f'(z)}{|f(z) - a|} \leq 4$. \hfill $\square$

## 2 $B$ and $B_0$ as Banach spaces

Recall $B = \{ f : f$ is a Bloch function $\}$. From (5),

$$\max_{|z| \leq r} |f(z)| \leq \left\{ 1 + \frac{1}{2} \log\left( \frac{1 + r}{1 - r} \right) \right\} ||f||, \quad (0 \leq r < 1)$$
Therefore choose $\delta < |\lambda|$. Hence $\lim_{r \to 1} ||f(rz)|| = ||f||$.

The functions

$$f_t(z) = \frac{e^{-it}}{2} \log \left( \frac{1 + ze^{-it}}{1 - ze^{-it}} \right)$$

for $0 \leq t < 2\pi$ are unbounded Bloch functions (the case $t = 0$ was seen in an example in Section 1.2). Note $||f_t - f_r||_\infty \geq 1$ for $t \neq r$, hence $\mathcal{B}$ is not separable (i.e. does not have a countable dense subset).

**Lemma 2.1.** The following are equivalent:

(i) $f_n \in \mathcal{B}_0$, $f \in \mathcal{B}$ and $||f_n - f|| \to 0$

(ii) The following two properties hold:

a) $f_n(z) \to f(z)$ as $n \to \infty$ locally uniformly in $\mathbb{D}$.

b) $(1 - |z|^2)|f_n'(z)| \to 0$ as $|z| \to 1$ uniformly in $n$.

**Proof.** (i)$\Rightarrow$(ii)

Let $f_n \in \mathcal{B}_0$, $f \in \mathcal{B}$ and $||f_n - f|| \to 0$. Then

$$(1 - |z|^2)|f_n'(z)| \leq |1 - |z|^2| |f_n'(z)| + ||f_n - f_n||$$

for $m, n > N(\epsilon)$ constant and $|z| < 1$. For some $\delta < 1$, $(1 - |z|^2)|f_n'(z)| < 2\epsilon$ for $n > N(\epsilon)$ and $\delta < |z| < 1$. Hence b) holds. a) follows from the observation above that convergence in the Bloch norm implies locally uniform convergence.

(ii)$\Rightarrow$(i)

$f_n \in \mathcal{B}_0$ by b). Also $f_n'(z) \to f'(z)$ for each $z \in \mathbb{D}$ by a). Thus $f \in \mathcal{B}_0$.

Therefore choose $\delta < 1$ such that $(1 - |z|^2)|f_n'(z) - f'(z)| < \epsilon$ for $n = 1, 2, ...$ and $\delta < |z| < 1$. Using a) to estimate the difference $|f_n'(z) - f'(z)|$ for $|z| \leq \delta$ implies $||f_n - f|| \to 0$.

**Theorem 2.2.** $\mathcal{B}_0$ is a separable closed nowhere dense subspace of $\mathcal{B}$ and is identical with the closure of the polynomials in the Bloch norm. Further $f \in \mathcal{B}_0$ if and only if $||f(z) - f(z\xi)|| \to 0$ as $\xi \to 1$, for $|\xi| \leq 1$.

**Proof.** From Lemma 2.1, $f_n \in \mathcal{B}_0$, $||f_n - f|| \to 0 \Rightarrow f \in \mathcal{B}_0$. Thus $\mathcal{B}_0$ is closed.

Since every polynomial is in $\mathcal{B}_0$, so is the closure of the polynomials. Further, if $f \in \mathcal{B}$, then $f(z\xi) \in \mathcal{B}_0$ for every $\xi \in \mathbb{D}$. $\mathcal{B}_0$ is closed so $f \in \mathcal{B}_0$ because $||f(z) - f(z\xi)|| \to 0$.

Conversely, let $f \in \mathcal{B}_0$ and $\xi_n \to 1$, $|\xi_n| \leq 1$. $f(\xi_n z)$ in $\mathcal{B}_0$ satisfies a) and b) in Lemma 2.1 thus $||f(z) - f(z\xi)|| \to 0$. Choose $\xi_n = 1 - \frac{1}{n}$, then $f(\xi_n z)$ is analytic in $|z| \leq 1$. Hence $\exists$ polynomial $p_n(z)$ such that $|f_n(\xi_n z) - p_n(z)| < \frac{1}{n}$
for $|z| < 1$. Thus $||f - p_n|| \leq ||f(z) - f(\xi_n z)|| + \frac{2}{n} \to 0$. So $B_0$ is identical with the closure of the polynomials and hence $B_0$ is separable.

If $f_0 \in B_0$ and $\epsilon > 0$, then $f(z) = f_0(z) + \frac{2}{n} \log \left( \frac{1 + z}{1 - z} \right) g(z)$ for $g \in B$ and $||g|| \leq \frac{\epsilon}{2}$ satisfies $||f - f_0|| < \epsilon$ and $f \notin B_0$. Thus $B_0$ is nowhere dense.

To investigate the dual spaces of $B_0$ and $B$, need to introduce the space $I$, where $I$ is the $C$-linear space of all functions $g(z) = \sum_{n=0}^{\infty} b_n z^n$ analytic in $D$ for which

$$||g||_I = |g(0)| + \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |g'(re^{i\theta})| \, dr \, d\theta < \infty$$

From the definition of $||g||_I$ and the Cauchy integral formula,

$$\max_{|z| \leq r} |g(z)| \leq \frac{1}{1 - r} ||g||_I, \quad 0 \leq r < 1.$$  

As for $B_0$, can characterise strong convergence in $I$ by the following two conditions:

a) $g_n(z) \to g(z)$ as $n \to \infty$, locally uniformly in $D$.

b) $\int_D |g_n'(z)| \, dA(z) \to 0$ as $|z| \uparrow 1$, uniformly in $n$.

Here $A$ is area measure on $D$.

Theorem 2.2 has an analogue for $I$. The proof is similar to that for Theorem 2.2 and is omitted.

**Theorem 2.3.** $I$ is a separable Banach space. The polynomials are dense in $I$. Further $||g(z) - g(\zeta z)||_I \to 0$ as $\zeta \to 1$, $|\zeta| \leq 1$.

**Theorem 2.4.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in B$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in I$. Then $h(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ is continuous in $|z| \leq 1$ and, further, $|h(z)| \leq 2(||f(0)|| + ||f||_B) ||g||_I$. In particular,

$$< f, g > = \lim_{\delta \downarrow 1} \sum_{n=0}^{\infty} a_n b_n \delta^n = \lim_{\delta \downarrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(\delta e^{-i\theta}) g(\delta e^{i\theta}) d\theta$$

exists for $f \in B$, $g \in I$. Note that the sum need not converge for $\delta = 1$.

**Proof.** The Theorem follows by making suitable estimates on $|h(z)|$. The full proof can be seen in [2].
Theorem 2.5. a) For every $\psi \in \mathcal{I}^*$, the dual space of $\mathcal{I}$, $\exists$ unique $f \in \mathcal{B}$ such that
\[
\psi(g) = \langle f, g \rangle = \lim_{\delta \to 1} \sum_{n=0}^{\infty} a_n b_n \delta^n, \text{ for } g \in \mathcal{I}.
\] (6)

Conversely, this defines a bounded linear functional on $\mathcal{I}$ for each $f \in \mathcal{B}$. Also
\[
\frac{1}{3} (|f(0)| + ||f||_{\mathcal{B}}) \leq ||\psi||_{\mathcal{I}^*} \leq 2 (|f(0)| + ||f||_{\mathcal{B}}) \quad \text{(7)}
\]

b) For every $\phi \in \mathcal{B}_0^*$, $\exists$ unique $g \in \mathcal{I}$ such that
\[
\phi(f) = \langle f, g \rangle, \text{ for } f \in \mathcal{B}_0.
\] (8)

Conversely, this defines a bounded linear functional on $\mathcal{B}$ for each $g \in \mathcal{I}$. Also
\[
\frac{1}{3} ||g||_{\mathcal{I}} \leq ||\phi||_{\mathcal{B}_0^*} \leq 2 ||g||_{\mathcal{I}}. \quad \text{(9)}
\]

Proof. See [2].

The relations (6) and (8) define isomorphisms between $\mathcal{I}^*$ and $\mathcal{B}$, and between $\mathcal{B}_0^*$ and $\mathcal{I}$ respectively. The isomorphisms are continuous by (7) and (9) respectively, although not isometric.

(8) actually defines a bounded functional on the whole of $\mathcal{B}$, not just $\mathcal{B}_0$.

However not all functionals on $\mathcal{B}$ can be represented in this form since the Hahn-Banach Theorem implies the existence of non-zero functionals on $\mathcal{B}$ that vanish on $\mathcal{B}_0$.

The above isomorphisms show that under the canonical embedding $\mathcal{B} \subset \mathcal{B}^{**}$, $\mathcal{B} = \mathcal{B}_0^*$. So $\mathcal{B}$ is not a reflexive space. It can be shown, [2], that the cardinality of $\mathcal{B}^{**}$ is greater than that of $\mathcal{B}$, hence there are no mappings from $\mathcal{B}$ onto $\mathcal{B}^{**}$.

3 Behaviour of Bloch Functions

3.1 A non-linear maximum and minimum principle for Bloch functions

Theorem 3.1. Let $C$ be a circle or line with $C \cap \mathbb{T} \neq \emptyset$ and $G$ a domain with $G \subset \mathbb{D} \setminus C$, $A := \partial G \setminus C \subset \mathbb{D}$, $\gamma$ the angle between $C$ and $\mathbb{T}$ towards $G$.

If $f \in \mathcal{B}$ and $c = \frac{\gamma}{2 \sin \gamma} ||f||_{\mathcal{B}}$ for $0 \leq \gamma < \pi$, then
\( |f(z)| \leq a \leq c \) for \( z \in A \Rightarrow |f(z)| \leq \frac{2c}{\log(\frac{c}{a}) + 1} \) for \( z \in G \).

(ii) \( |f(z)| \geq a \geq ec \) for \( z \in A \Rightarrow |f(z)| \geq e^{-1}a \) for \( z \in G \), where \( e = \exp(1) \).

(iii) \( \text{dist}(f(A), f(G)) \leq ec \) for \( z \in G \).

Note that if \( \gamma = 0 \) then \( C \) touches \( T \) and \( c = \frac{|f|_{\infty}}{2} \). The important point is that we only need to know a bound for \( |f(z)| \) on \( A \) and not on \( C \). To see this, transform the disc to the annulus via the covering map

\[
\Phi(z) = \exp \left( i c \log \left( \frac{z - z_0}{1 - \bar{z}_1 z} \right) \right)
\]

where \( C \) intersects \( T \) at \( z_0 \) and \( z_1 \). Hadamard’s Three Circles Theorem then says that it can be arranged so that \( |f(z)| \) is as small as desired on \( \Phi(C) \).

Proof. a) Consider first the case \( \gamma > 0 \) and \( \partial G \subset \mathbb{D} \).

Consider the increasing function \( \varphi(x) = x \log(\frac{x}{a}) \) for \( e^{-1}a \leq x < +\infty \).

\( \varphi(e^{-1}a) = -e^{-1}a, \varphi(a) = 0 \) and \( \varphi(+\infty) = +\infty \).

So for each \( a > 0, \exists b > 0 \) such that \( \varphi(b) = c \) and if \( c \leq e^{-1}a \), then \( \exists b' \) with \( e^{-1}a \leq b' \leq a \) such that \( \varphi(b') = -c \).

Without loss of generality assume \( \pm i \in C \) and \( G \) lies to the right of \( C \). If not, replace \( f \) by \( f \circ \tau \) for a suitable Möbius transformation \( \tau \). For \( 0 < \gamma' < \gamma \), let \( G' \) denote the open subset of \( G \) cut off by the circle \( C' \) through \( \pm i \) that forms the angle \( \gamma' \) with \( T \).

(i) \( |f(z)| \leq a \leq c \) for \( z \in A = \partial G \setminus C \). Claim that \( |f(z)| \leq b \) for \( z \in G \).

Suppose not for a contradiction. Since \( a < b \) and \( f \) continuous in \( \overline{G} \subset \mathbb{D} \),

\( \exists \gamma' < \gamma \) such that \( |f(z)| \leq b = |f(x_0)| \) for \( z \in G' \) with \( x_0 \in C' \cap \partial G' \). (10)
Without loss of generality, $x_0 \in \mathbb{R}$, otherwise apply a suitable Möbius transformation. $(x_0, x_0 + \delta) \subset G'$ for some $\delta > 0$, otherwise $|f(x_0)| \leq a < b$.

Let $u(z) = \arg(i + z) - \arg(1 + iz)$ for $z \in \mathbb{D}$. Then $u(z) > 0$ for $z \in \mathbb{D}$. $u(z) = \gamma'$ for $z \in C'$. $u(x) = \frac{x}{2} - 2 \arctan(x)$ for $x \in \mathbb{R}$.

Thus $v(z) := \frac{au(z)}{b\gamma'} - \log(|f(z)|)$ satisfies $v(z) \geq - \log(a)$ for $z \in \partial G' \setminus C' \subset \partial G \setminus C$, since $|f(z)| \leq a$.

(10) $\Rightarrow v(z) \geq \frac{x}{2} - \log(b) = - \log(a)$ for $z \in \partial G' \cap C'$ since $u(z) = \gamma'$ for $z \in C'$.

There is equality for $z = x_0$. $v$ is harmonic except for positive logarithmic poles. The minimum principle for harmonic functions implies $v(z) \geq - \log(a)$ for $z \in G'$. Therefore, $v(x) \geq - \log(a) = v(x_0)$ for $x_0 < x < x_0 + \delta$.

\[
0 \leq v'(x_0) = \frac{-2c}{b\gamma'(1 + x_0^2)} - \text{Re} \frac{f'(x_0)}{f(x_0)}, \quad \text{(since } w'(x) = \frac{-2}{1 + x^2} \text{ for } x \in \mathbb{R})
\]

\[
\leq \frac{-2c}{b\gamma'(1 + x_0^2)} + \frac{|f||g|}{b(1 + x_0^2)} \quad \text{(since } |f(z)| \leq \frac{|f||g|}{1 - |z|} \forall z \in \mathbb{D}).
\]

Rearranging gives

\[
\frac{\gamma||f||g}{2 \sin \gamma} = \frac{\gamma'|f||g|}{2} \left(\frac{1 + x_0^2}{1 - x_0^2}\right) = \gamma'|f||g| \quad \frac{2 \sin \gamma}{\gamma'} \geq \frac{\gamma'|f||g|}{2} \left(\frac{1 + x_0^2}{1 - x_0^2}\right).
\]

This contradicts $\gamma' < \gamma$.

Hence $|f(z)| \leq b$ for $z \in G$. $c = \varphi(b) = b \log(\frac{a}{e})$.

\[
\frac{1}{b} = \frac{1}{c} \log \left(\frac{b}{a}\right) \geq \frac{1}{2c} \left(\log \left(\frac{b}{a}\right) + \log \left(\frac{b}{a}\right) + 1\right) = \frac{1}{2c} \left(\log \left(\frac{c}{a}\right) + 1\right) > 0
\]

since $a < c$. Hence

\[
|f(z)| \leq \frac{2c}{\log \left(\frac{c}{a}\right) + 1}
\]

(ii) Let $|f(z)| \geq a \geq ce$ for $z \in \partial G \setminus C$. Claim that $|f(z)| \geq b'$ for $z \in G$. Since $b' < a$, there would otherwise exist $\gamma' < 0$ such that $|f(z)| \geq b' = |f(x_0)|$ for $z \in G'$ with $x_0 \in C' \cap \partial G'$. Again assume $x_0 \in \mathbb{R}$.

Since $\varphi(b') = -c$, the function $v(z) = c \frac{u(z)}{b\gamma'} + \log(|f(z)|)$ satisfies $v(z) \geq \log(a)$ for $z \in \partial G' \setminus C'$ and $z \in C'$. Hence $v(z) \geq \log(a)$ for $z \in G'$ by minimum principle. Thus $v'(x_0) \geq 0$ leading to a contradiction as above. Hence $|f(z)| \geq b' \geq e^{-1}a$ for $z \in G$.

(iii) $z \in G$ and suppose $\text{dist}(f(z), f(A)) \geq ce$. Then $|f(\xi) - f(z)| \geq ce$ for $\xi \in A$. Thus $|f(\xi) - f(z)| \geq c > 0$ for $\xi \in G$ by (ii), contradicting $z \in G$.

b) For the general case, let $C_n$ be circles with $C_n \cap G \neq \emptyset$ and $C_n \to C$. Let $G_n$ be the open subset of $G$ cut off by $C_n$. See the diagram on the following page.

Then $\partial G_n \subset \mathbb{D}$ and $\gamma_n > 0$, so can apply part a) to $G_n$. $\gamma_n \to \gamma$, so the assertions follow as $n \to \infty$. 

10
3.2 Boundary Behaviour

The first theorem in this section shows that for a Bloch function, every asymptotic value is an angular limit.

**Theorem 3.2.** Let \( \Gamma \) be a Jordan arc in \( \mathbb{D} \) ending at \( \xi \in \mathbb{T} \). If \( f \in \mathcal{B} \) and if \( f(z) \to b \in \mathbb{C}_\infty \) as \( z \to \xi \) with \( z \in \Gamma \), then \( f \) has angular limit \( b \) at \( \xi \).

**Proof.** The proof uses Theorem 3.1.

Let \( \Delta \) be a Stolz angle at \( \xi \). Assume \( b \neq \infty \) and let \( \frac{3\pi}{4} < \gamma < \pi \) and define \( c = \frac{\gamma}{2\sin\gamma} ||f||_B \). Let \( \Gamma^* \) be the subarc of \( \Gamma \) ending at \( \xi \) such that \( |f(z) - b| < \epsilon < c \) for \( z \in \Gamma^* \).

(11) and Theorem 3.1(i) \( \Rightarrow |f(z) - b| < \frac{2c}{\log \left( \frac{z}{\gamma} \right) + 1} \) for \( z \in G^\pm \) and thus for \( z \in \Delta^* \).

\[
\frac{2c}{\log \left( \frac{z}{\gamma} \right) + 1} \to 0 \text{ as } \epsilon \to 0
\]

so \( f \) has angular limit \( b \) at \( \xi \).

The case \( b = \infty \) is handled in the same way using Theorem 3.1(ii).

**Proposition 3.3.** Let \( \Gamma \) be a Jordan arc in \( \mathbb{D} \) ending at \( \xi \in \mathbb{T} \) and let \( f \in \mathcal{B} \). For \( 0 < \alpha < \frac{\pi}{2} \), \( \exists K = K(\alpha) \) such that \( \text{dist}(f(z), f(\Gamma)) \leq K||f||_B \) for \( z \in \Delta \), where \( \Delta = \{ |\arg(1 - \xi z)| < \alpha, |z - \xi| < \rho \} \) with suitable \( \rho > 0 \).

**Proof.** The proof follows from Theorem 3.1(iii) and the construction used in the proof of Theorem 3.2 and is thus omitted.
If $\Gamma$ approaches $\xi$ non-tangentially, so that some subset of $\Gamma$, say $\tilde{\Gamma}$, is eventually contained in $\Delta$ for suitable $\Delta$, then for $w \in \tilde{\Gamma}$, there is a $z \in \Delta$ such that $\rho(z, w) < K$ for some $K$, and $\rho$ is the hyperbolic metric. Then, as in Proposition 3.3,
\[
|f(z) - f(w)| \leq ||f||B\rho(z, w) < K||f||B.
\]
In particular, if $|f|$ is bounded on $\Gamma$, then $|f|$ is bounded on $\Delta$ for $f \in B$.

Also, $\text{Re } f \to \pm \infty$ on $\Gamma \Rightarrow \text{Re } f$ has angular limit $\pm \infty$ at $\xi$.

Theorem 3.2 can be extended to show that the asymptotic values of non-zero analytic functions on $\mathbb{D}$ are angular limits in the following way.

**Corollary 3.4.** If $h$ is analytic and non-zero in $\mathbb{D}$, $\log(h) \in B$ and if $h$ has asymptotic value $b \in \mathbb{C}_\infty$ at $\xi \in T$, then $h$ has angular limit $b$ at $\xi$.

**Proof.** The case $b \neq 0, \infty$ follows from Theorem 3.2. If $b = 0$, $\text{Re } \log(h)$ has asymptotic value $-\infty$ at $\xi$ and thus the angular limit $-\infty$ by Proposition 3.3. Thus $h$ has angular limit $0$ at $\xi$.

If $b = \infty$, $\text{Re } \log(h)$ has asymptotic value $\infty$ at $\xi$ and again by Proposition 3.3, angular limit $\infty$ at $\xi$. Thus $h$ has angular limit $\infty$ at $\xi$.

Another application of Theorem 3.1 is the following.

**Proposition 3.5.** Let $J$ be an arc of $T$ and $f \in B$. Then $\exists \xi \in J$ such that $|f(r\xi) - f(0)| \leq \frac{2||f||B}{\Lambda(J)}$ for $0 \leq r < 1$. Here, $\Lambda(J)$ is the length of $J$.

**Proof.** Let $\theta = \frac{\Lambda(J)}{2}$. Assume $f(0) = 0$, $||f||B = 1$ and $J$ is the arc of $T$ between $e^{-i\theta}$ and $e^{i\theta}$. (Can do this via a Möbius transformation, scaling and rotation respectively)

The circle $C$ through $e^{i\theta}, e^{-i\theta}$ and $0$ forms an inner angle $\gamma = \pi - \theta$ with $T$ at $e^{\pm i\theta}$. Let $G$ be the component of $\{z \in C : |f(z)| < a = \frac{e^\theta}{2\theta}\}$ with $0 \in \partial G$.

Claim that $J \cap \partial G \neq \phi$. Since $a \geq cc$ where $c = \frac{\gamma}{2\sin \gamma}||f||B$, it would otherwise mean from Theorem 3.1(ii) that $0 = |f(0)| \geq e^{-\theta}a > 0$.

Thus $\exists z_n \in G$ with $z_n \to \xi \in J \cap \partial G$ as $n \to \infty$. Since $0$ and $z_n$ can be connected by a curve $P_n \subset G$, it follows from Theorem 3.1(iii) with $\gamma = \frac{\pi}{2}$, $c = \frac{\pi}{2}$ that $|g(rz_n)| \leq \frac{e^\theta}{2\theta} + \frac{e^\theta}{2\theta} < \frac{11}{10}$ for $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. Let $n \to \infty$. \qed
3.3 Integrals of Bloch Functions

For this section, define
\[ h(z) = \int_0^z \frac{f(\xi) - f(0)}{\xi} d\xi, \quad z \in \mathbb{D}, \quad f \in B. \]

**Proposition 3.6.** If \( f \in B \), then \( h \) is continuous in \( \mathbb{D} \) and for \( |z| \leq 1 \) and \( 0 < |t| \leq \pi \),
\[ \left| \frac{h(e^{it}z) - h(z)}{it} - f \left( 1 - \frac{|t|}{\pi} \right) z + f(0) \right| \leq K||f||_B \]
for \( K \) an absolute constant.

**Proof.** \( h'(z) = O((1 - |z|)^{-\frac{3}{2}}) \) as \( |z| \to 1 \), so \( h \) is continuous in \( \mathbb{D} \). Without loss of generality, \( f(0) = 0, \frac{1}{2} < |z| < 1 \) and \( 0 < |t| < \frac{\pi}{2} \). For \( 0 < t < \frac{\pi}{2} \), let \( |\tau| \leq t \) and choose \( \sigma \) such that \( t = \pi(1 - e^{-\sigma}) \). Integration by parts shows that
\[ \int_{e^{-\sigma}z}^{e^{\tau}z} f'(\xi) \log \left( \frac{e^{i\tau}z}{\xi} \right) d\xi = h(e^{i\tau}z) - h(e^{-\sigma}z) - (\sigma + i\tau)f(e^{-\sigma}z) \]
Also,
\[ \left| \log \left( \frac{e^{i\tau}z}{\xi} \right) \right| \leq \log \left( \frac{1}{|\xi|} \right) + \left| \arg \left( \frac{e^{i\tau}z}{\xi} \right) \right| \leq K'(1 - |\xi|) \]
for \( K' \) absolute constant and for \( \xi \in [e^{-\sigma}z, e^{i\tau}z] \). Thus the integrand is bounded by \( K'||f||_B \). Hence
\[ |h(e^{i\tau}z) - h(e^{-\sigma}z) - (\sigma + i\tau)f(e^{-\sigma}z)| \leq K''|\sigma||f||_B \]
The result follows by choosing \( \tau = \pm t \) and \( \tau = 0 \) and then subtracting. \( \square \)

The following theorem provides another characterisation of Bloch functions.

**Theorem 3.7.** Let \( f \) be analytic in \( \mathbb{D} \). Then \( f \in B \) if and only if \( h \) as defined above is continuous in \( \mathbb{D} \) and
\[ \max_{|z|=1} |h(e^{it}z) + h(e^{-it}z) - 2h(z)| \leq Mt \quad (12) \]
for \( t > 0 \) and \( M \) constant.

**Proof.** Let \( f \in B \). Using Proposition 3.6 with \( t, -t \) and then subtracting gives
\[ ||h(e^{it}z) + h(e^{-it}z) - 2h(z)|| \leq 2K|t||f||_B \]
Conversely, let (12) hold. Since \( h \) analytic in \( \mathbb{D} \) and continuous \( D \cup T \), the Poisson integral formula shows that for \( 0 < r < 1 \),
\[ h(re^{is}) - h(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(s - \tau) + r^2}(h(e^{i\tau}) - h(e^{i\theta}))d\tau \]
Differentiating twice with respect to $s$ and substituting $s = \theta$, $\tau = \theta \pm t$ gives

$$e^{i\theta} f'(re^{i\theta}) =$$

$$\frac{1 - r^2}{\pi} \int_0^\pi \frac{(1 + r^2) \cos(t) - 2r - 2r\sin(t))^2}{(1 - 2r \cos(t) + r^2)^3} [h(e^{i(\theta+t)}) + h(e^{i(\theta-t)}) - 2h(e^{i\theta})] dt$$

The inner quotient is bounded by $C[t(1 - r)(1 - 2r \cos(t) + r^2)]^{-1}$ for $C$ constant. Hence

$$(1 - r^2) |f'(re^{i\theta})| \leq \frac{2CM}{\pi} \int_0^\pi \frac{1 - r^2}{1 - 2r \cos(t) + r^2} dt = 2CM$$

Hence $f \in B$. \qed

A continuous function $h : \mathbb{T} \to \mathbb{C}$ belongs to the Zygmund class $\Lambda_\ast$ if (12) is satisfied. Thus if $h$ analytic in $D$, $h \in \Lambda_\ast \Leftrightarrow h' \in B$.

Actually, assuming (12) with $h$ replaced by $\text{Im } h$, then $(1 - r^2)|\text{Im}(e^{i\theta} f'(re^{i\theta}))| \leq C'M$ for $C'$ constant, and this is enough to conclude that $f \in B$.

A non-negative finite measure $\mu$ on $\mathbb{T}$ is called a Zygmund measure if

$$|\mu(J) - \mu(J')| \leq M' \Lambda(J)$$

for $M'$ constant, and $J$, $J'$ are adjacent arcs of equal length. Hence the corresponding increasing function

$$\mu^*(t) = \mu(\{e^{i\tau} : 0 \leq \tau \leq t\})$$

is continuous and

$$|\mu^*(\theta + t) + \mu^*(\theta - t) - 2\mu^*(\theta)| \leq M't$$

for $t > 0$. Such measures can be used to characterise Bloch functions of positive real part.

**Proposition 3.8.** $f \in B$ and $\text{Re } f(z) > 0$ for $z \in D$ if and only if it can be represented as

$$f(z) = ib + \int_\mathbb{T} \frac{\xi + z}{\xi - z} d\mu(\xi)$$

for $z \in D$, $\mu$ a Zygmund measure and $b \in \mathbb{R}$.

**Proof.** Proof omitted. See [1] for details. \qed

The integral in Proposition 3.8 is in fact the singular inner factor associated with the positive singular measure $\mu$. Inner functions will be met in Section 5.
3.4 Growth of Bloch Functions

Hardy’s identity for $f$ analytic in $D$ is

$$\frac{d}{dr} \left( r \frac{d}{dr} \int_0^{2\pi} |f(re^{it})|^p \, dt \right) = p^2 r \int_0^{2\pi} |f(re^{it})|^p - 2 \frac{d}{dt} |f(re^{it})|^2 \, dt \quad (13)$$

The next theorem gives a bound on the integral means of Bloch functions.

**Theorem 3.9.** If $f \in B$ and $f(0) = 0$, then

$$\frac{1}{2\pi} \int_T |f(r\xi)|^{2n} |d\xi| \leq n!||f||^2_B \left( \log \left( \frac{1}{1-r^2} \right) \right)^n$$

for $0 < r < 1$ and $n \in \mathbb{N}$.

**Proof.** The proof will use induction. The case $n = 0$ is immediate. Suppose the result holds for some $n$, then Hardy’s identity (13) shows that

$$\frac{d}{dr} \left( r \frac{d}{dr} \int_0^{2\pi} |f(r\xi)|^{2n+2} |d\xi| \right) = 4(n+1) \frac{1}{2\pi} r \int_0^{2\pi} |f'(r\xi)|^{2n} |f''(r\xi)|^2 |d\xi|$$

$$\leq 4(n+1) n! ||f||^2_B 2n \lambda(r) (1-r^2)^{-2} ||f||^2_B, \text{ writing } \lambda(r) = \log \left( \frac{1}{1-r^2} \right)$$

$$\leq (n+1) n! ||f||^2_B 2n+2 \frac{d}{dr} \left( r \frac{d}{dr} \lambda(r)^{n+1} \right)$$

Integrating,

$$\frac{d}{dr} \left( \frac{1}{2\pi} \int_T |f(r\xi)|^{2n+1} |d\xi| \right) \leq (n+1) n! ||f||^2_B 2^{n+1} \frac{d}{dr} (\lambda(r)^{n+1})$$

and the result follows for $n+1$ by integrating again, since both sides vanish for $r=0$.

Note that as $r \to 1$,

$$\log \left( \frac{1}{1-r^2} \right) \sim \log \left( \frac{1+r}{1-r} \right) = \rho(0, r).$$

So Theorem 3.9 can be rephrased in terms of the hyperbolic metric,

$$\frac{1}{2\pi} \int_T |f(r\xi)|^{2n} |d\xi| \leq n! ||f||^2_B \rho(0, r)^n.$$

In fact, a slightly different inequality can be obtained using (5),

$$\frac{1}{2\pi} \int_T |f(r\xi) - f(0)|^{2n} |d\xi| \leq \frac{1}{2\pi} \int_T ||f||^2_B 2^n \rho(r\xi, 0)^{2n} |d\xi| = ||f||^2_B 2^n \rho(r, 0)^{2n}.$$
Theorem 3.10 (Makarov law of the iterated logarithm). If \( f \in B \), then for almost all \( \xi \in \mathbb{T} \),
\[
\limsup_{r \to 1} \frac{|f(r\xi)|}{\sqrt{\log \left( \frac{1}{1-r} \right) \log \log \left( \frac{1}{1-r} \right)}} \leq \|f\|_B
\]

Proof. Without loss of generality, \( \|f\|_B = 1 \). Define
\[
f^*(s, \xi) = \max_{0 \leq r \leq 1-e^{-s}} |f(r\xi)|, \quad \text{for } e \leq s < \infty, \xi \in \mathbb{T}.
\]

The Hardy-Littlewood maximal theorem applied to the analytic function \( f^{2n} \) and Theorem 3.9 show that
\[
\int_{\mathbb{T}} f^*(s, \xi)^{2n} |d\xi| \leq \frac{K}{2\pi} \int_{\mathbb{T}} |f((1-e^{-s})\xi)|^{2n} |d\xi| \leq n!Ks^n
\]
where \( K \) is an absolute constant. Multiply by \( s^{-n} \psi_n(s) \), where
\[
\psi_n(s) = -n \frac{d}{ds} \left( \log(s)^{-\frac{1}{n}} \right) = s^{-1}(\log(s))^{-1-\frac{1}{n}} > 0
\]
Integrating and using Fubini’s theorem,
\[
\int_{\mathbb{T}} \left( \int_e^{\infty} f^*(s, \xi)^{2n} s^{-n} \psi_n(s) |d\xi| \right) ds \leq n!nK
\]
Hence \( \exists \) sets \( A_n \subset \mathbb{T} \) with \( \Lambda(A_n) > 2\pi - \frac{K}{n} \) such that
\[
\int_e^{\infty} f^*(s, \xi)^{2n} s^{-n} \psi_n(s) ds \leq n!n^4, \quad \text{for } \xi \in A_n.
\]
Since
\[
-\frac{d}{ds} \left( s^{-n}(\log(s))^{-1-\frac{1}{n}} \right) \leq 3ns^{-n} \psi_n(s),
\]
for \( \xi \in A_n \) and \( e \leq s < \infty \),
\[
f^*(s, \xi)^{2n} \sigma^{-n}(\log(\sigma))^{-1-\frac{1}{n}} \leq 3n \int_{\sigma}^{\infty} f^*(s, \xi)^{2n} s^{-n} \psi_n(s) ds \leq 3n!n^4.
\]
The set
\[
A = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n
\]
satisfies \( \Lambda(A) = 2\pi \) because \( \Lambda(A_n) > 2\pi - \frac{K}{n} \). Let \( \xi \in A \), then \( \xi \in A_n \) for \( n \geq k \) and suitable \( k = k(\xi) \). If \( r < 1 \) is close enough to 1, \( n = \log \log(\sigma) \geq k \), where \( \sigma = \log \left( \frac{1}{1-r} \right) \). Since \( \log \log(\sigma) \geq n \) and \( \log(\sigma) < e^{n+1} \),
\[
\frac{|f(r\xi)|^2}{\sigma \log \log(\sigma)} \leq \frac{f^*(s, \xi)^2}{\sigma \log \log(\sigma)} \leq \left( \frac{3n!n^4}{n^n} \right)^{\frac{1}{n}} e^{-\frac{(n+1)^2}{n^2}} \to 1
\]
as \( n \to \infty \) by Stirling. \( \square \)
Makarov’s theorem has a probabilistic interpretation. Consider the probability measure \((2\pi)^{-1} \Lambda\) on \(T\). Then \(Z_t(\xi) = f((1-e^{-t})\xi)\) for \(\xi \in T\) and \(0 \leq t < \infty\) defines a complex stochastic process with expectation \(f(0)\). Its variance satisfies \(\sigma(t)^2 \leq ||f||_B^2 t\) by Theorem 3.9 with \(n = 1\). The result of Theorem 3.10 can be rewritten as

\[
\limsup_{t \to \infty} \frac{|Z_t|}{||f||_B \sqrt{t \log(t) \log(t)}} \leq 1.
\]

This is a one-sided form of the law of the iterated logarithm.

**Corollary 3.11.** Let \(f : D \to \mathbb{C}\) be univalent. Then

\[
\limsup_{r \to 1} \frac{|\log(f'(r\xi))|}{\sqrt{\log \left(\frac{1}{1-r}\right) \log \log \left(\frac{1}{1-r}\right)}} \leq 6
\]

for almost all \(\xi \in T\). In particular, \(f'(r\xi) = O((1-r)^{-\epsilon})\) as \(r \to 1\) for \(\epsilon > 0\) and almost all \(\xi\).

**Proof.** From Theorem 3.10 and Proposition 1.1. \(\square\)

### 4 Composition Operators on \(B\)

#### 4.1 Composition Operators with closed range

The pseudohyperbolic distance on \(D\) is defined by

\[
\tilde{\rho}(z, w) = \left|\frac{z - w}{1 - \overline{z}w}\right|
\]

for \(z, w \in D\), so that \(\rho(z, w) = \frac{1}{2} \tanh^{-1}(\tilde{\rho}(z, w))\). Also define

\[
\varphi_{\xi}(w) = \frac{z - w}{1 - \overline{z}w}.
\]

For \(\varphi : D \to D\) holomorphic, the composition operator \(C_{\varphi} : H(D) \to H(D)\) is defined by \(C_{\varphi}f = f \circ \varphi\), where \(H(D) = \{\text{analytic functions on } D\}\). In this section, \(G = \varphi(D)\) and

\[
\tau_{\varphi}(z) = \frac{(1 - |z|^2)\varphi'(z)}{1 - |\varphi(z)|^2
\]

and \(||.||\) denotes the norm on \(B\), ie. \(||f|| = |f(0)| + ||f||_B\).

The Schwarz-Pick Lemma implies

(i) \(C_{\varphi}\) maps \(B\) to \(B\).

(ii) \(0 \leq |\tau_{\varphi}(z)| \leq 1\) \hspace{1cm} (14)

(iii) \(\varphi \in B_0\) if \(C_{\varphi}\) maps \(B_0 \to B_0\) and conversely, \(f, \varphi \in B_0\) \(\Rightarrow f \circ \varphi \in B_0\).
Lemma 4.1. \( C_{\varphi} : \mathcal{B} \to \mathcal{B} \) is bounded below if and only if \( C_{\varphi} \) has closed range.

Proof. Assume \( \|C_{\varphi}f\| \geq \epsilon\|f\| \) for some \( \epsilon > 0 \) and \( \forall f \in \mathcal{B} \). Suppose \((f_n)\) sequence of functions in \( \mathcal{B} \) with \( C_{\varphi}f_n \to g \in H(\mathbb{D}) \).

Now \( \|f_n - f_m\| \leq \epsilon^{-1}\|C_{\varphi}f_n - C_{\varphi}f_m\| \), so \((f_n)\) is Cauchy and hence converges to \( f \in \mathcal{B} \). Then

\[
C_{\varphi}f = \lim_{n \to \infty} C_{\varphi}f_n = g \in C_{\varphi}(\mathcal{B})
\]

since \( C_{\varphi} \) continuous. Thus \( C_{\varphi} \) has closed range.

Conversely, for every \( f \in \mathcal{B} \), there is a sequence \((f_n)\) with \( f_n \to f \) eg.

\( f_n(z) = (1 - \frac{1}{n})f(z) \). \( C_{\varphi}f_n \to C_{\varphi}f \) in \( \mathcal{B} \) for \( \varphi \) holomorphic and assumed non-constant. Write \( g_n = C_{\varphi}f_n \) and \( g = C_{\varphi}f \). Now \( k_n\|f_n\| = \|g_n\| \) for some \( k_n \in \mathbb{R}, \forall n \). Let \( K = \lim \inf_{n \to \infty}(k_n) \). Then \( \|g_n\| \geq K\|f_n\| \).

If \( K = 0 \), for a contradiction, then, by passing to a subsequence if necessary, \( k_n\|f_n\| \to 0 \) as \( j \to \infty \), ie. \( \|g_n\| \to 0 \). But \( g_n \to g \) so \( \|g\| = 0 \).

Thus \( g \equiv 0 \), so \( f(\varphi(z)) = 0 \), \( \forall z \in \mathbb{D} \). \( f \) is non-constant holomorphic, so the image of \( \varphi \) must be contained in the kernel of \( f \), but by the principle of isolated zeros, \( \varphi \) must be constant, giving a contradiction.

Therefore \( K > 0 \) and \( \|C_{\varphi}f\| \geq K\|f\| \). \( \Box \)

So the closed range condition of \( C_{\varphi} \) is equivalent to the bounded below condition.

Recall \( \|f\|_B = \sup_{z \in \mathbb{D}} \{(1 - |z|^2)|f'(z)|\} \) whereas \( \|f\| = \|f\|_B + |f(0)| \).

Lemma 4.2. If \( C_{\varphi} \) bounded below on \( \mathcal{B} \), then \( \forall f \in \mathcal{B} \),

\[
\|f\|_B \leq k \sup_{w \in G}\{(1 - |w|^2)|f'(w)|\}
\]

for some \( k \).

Proof. The bounded below condition gives

\[
\|f\|_B \leq k\|C_{\varphi}f\|_B \leq k\sup_{z \in \mathbb{D}}\{(1 - |z|^2)|f'(\varphi(z))\varphi'(z)|\}
\]

\[
\leq k\sup_{z \in \mathbb{D}}\{(1 - |\varphi(z)|^2)|f'(\varphi(z))|\varphi'(z)|\}, \text{ by (14)}
\]

\[
\leq k\sup_{w \in G}\{(1 - |w|^2)|f'(w)|\}, \text{ since } |\varphi'(z)| \leq 1.
\]

\( \Box \)

Proposition 4.3. If \( C_{\varphi} \) is bounded below on \( \mathcal{B} \), then \( \exists \epsilon, r \) with \( r < 1 \) such that \( \forall z \in \mathbb{D} \), \( \rho(\varphi(z), z) \leq r \), where \( \Omega_r = \{z \in \mathbb{D} : |\varphi(z)| > \epsilon\} \).

Proof. Since \( C_{\varphi} : \mathcal{B} \to \mathcal{B} \) bounded below, \( \exists \) constant \( 0 < k \leq 1 \) such that \( \|C_{\varphi}f\| \geq k\|f\| \) for \( f \in \mathcal{B}_0 \). For each \( w \in \mathbb{D} \), let

\[
f_w(z) = \frac{w - z}{1 - \overline{w}z} - \frac{w - \varphi(0)}{1 - \overline{w}\varphi(0)}.
\]
$f_w$ is a bounded and continuous analytic function on $\mathbb{D}$ and so is in $B_0$. Moreover, $\|f_w\| \geq 1$. Thus $\|C_\varphi f_w\| \geq k\|f_w\| \geq k$. Also

$$(1 - |z|^2)(|C_\varphi f_w|^2(z)) = (1 - |\varphi_w(\varphi(z))|^2)|\tau_\varphi(z)|$$

and $(C_\varphi f_w)(0) = 0$. There is a point $z_w \in \mathbb{D}$ such that

$$(1 - |z_w|^2)(|C_\varphi f_w|^2(z_w)) \geq \frac{k}{2}.$$ 

So

$$(1 - |\varphi_w(\varphi(z_w))|^2)|\tau_\varphi(z_w)| \geq \frac{k}{2}.$$ 

Thus $(1 - |\varphi_w(\varphi(z_w))|^2) \geq \frac{k}{2}$, $|\tau_\varphi(z_w)| \geq \frac{k}{2}$ and so $|\varphi_w(\varphi(z_w))|^2 \leq 1 - \frac{k}{2}$.

Let $r = \sqrt{1 - \frac{k}{2}} < 1$ and $\epsilon = \frac{k}{2}$. $\tilde{\rho}(\varphi(z_w), w) = |\varphi_w(\varphi(z_w))|$, so $\tilde{\rho}(\varphi(z_w), w) < r$ and $|\tau_\varphi(z_w)| \geq \epsilon$.

Geometrically, $\Omega_\epsilon \subset \mathbb{D}$ and as $\epsilon \to 1$, $\Omega_\epsilon \to \phi$ and also as $\epsilon \to 0$, $\Omega_\epsilon \to \mathbb{D}$.

Let $r = \sqrt{1 - \frac{k}{2}} < 1$ and $\epsilon = \frac{k}{2}$.

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Geometrically, $\Omega_\epsilon \subset \mathbb{D}$ and as $\epsilon \to 1$, $\Omega_\epsilon \to \phi$ and also as $\epsilon \to 0$, $\Omega_\epsilon \to \mathbb{D}$.

Theorem 4.4. Let $f \in B$, then

$$\left| (1 - |z|^2)||f'(z)| - (1 - |w|^2)||f'(w)| \right| \leq K\tilde{\rho}(z, w)||f||_B$$

for $z, w \in \mathbb{D}$ and $K = 3.31$.

This will show that $(1 - |z|^2)||f'(z)|$ is Lipschitz for $f \in B$, and will lead on to the next theorem, which gives some sufficient conditions for $C_\varphi$ to have closed range. The proof of Theorem 4.4 will be postponed until the end of the section.

Theorem 4.5. If, for some constants $0 < r < \frac{1}{2}$ and $\epsilon > 0$, for each $w \in \mathbb{D}$, $\exists z_w \in \mathbb{D}$ such that $\tilde{\rho}(\varphi(z_w), w) < r$ and $|\tau_\varphi(z_w)| > \epsilon$, then $C_\varphi : B \to B$ is bounded below.
Proof. Let \( \lambda = \varphi(0) \). Then \( \varphi = \varphi \lambda \circ \varphi \lambda \). Define \( \psi = \varphi \lambda \circ \varphi \). Then \( \psi(0) = 0 \) and \( C \varphi = C \varphi \lambda \). \( \varphi \lambda \) is a Möbius transformation, so \( C \varphi \lambda \) is an isometry on \( B \).
So required to prove that \( C \psi \) is bounded below on \( B \), since \( \psi \) still satisfies the conditions of the theorem.

To prove \( C \psi \) bounded below on \( B \), sufficient to prove \( ||C \psi f|| \geq k \) for some \( k > 0 \) and \( \forall f \in B \) with \( ||f|| = 1 \).

For each \( z \in \mathbb{D} \), have

\[
(1 - |z|^2)|(C \psi f)'(z)| = (1 - |\psi(z)|^2)|f'(\psi(z))||\tau_\psi(z)|
\]
by (14). Since \( ||f|| = 1 \), \( \exists w \in \mathbb{D} \) such that

\[
(1 - |w|^2)|f'(w)| \geq \left( 1 - \frac{1}{2} - \frac{r}{2} \right) (1 - |f(0)|).
\]
By Theorem 4.4,

\[
|1 - |w|^2||f'(w)| - (1 - |z|^2)|f'(z)|| \leq 4\tilde{d}(z, w)(1 - |f(0)|).
\]
Thus whenever \( \tilde{d}(\psi(z), w) < r < \frac{1}{4} \),

\[
(1 - |\psi(z)|^2)|f'(\psi(z))| \geq (1 - |w|^2)|f'(w)| - 4r(1 - |f(0)|)
\]
\[
\geq \left( 1 - \frac{1}{2} - \frac{r}{2} \right) (1 - |f(0)|) = \frac{7(1 - 4r)}{8}(1 - |f(0)|)
\]
Therefore,

\[
||C \psi f|| \geq |f(0)| + (1 - |z|^2)|(C \psi f)'(z)|
\]
\[
\geq |f(0)| + (1 - |\psi(z)|^2)|f'(\psi(z))||\tau_\psi(z)|, \quad \forall z \in \mathbb{D}.
\]
In particular,

\[
||C \psi f|| \geq |f(0)| + (1 - |\psi(z)|^2)|f'(\psi(z))||\tau_\psi(z)|
\]
\[
\geq |f(0)| + \frac{7(1 - 4r)\epsilon}{8}(1 - |f(0)|) \geq \frac{7(1 - 4r)\epsilon}{8}
\]

Let \( k = \frac{7(1 - 4r)\epsilon}{8} \), then \( ||C \psi f|| \geq k \) when \( ||f|| = 1 \) as required.

\[\square\]

Proof of Theorem 4.4. For \( z, w \in \mathbb{D} \), let \( \lambda = \varphi_w(z) \). Then \( (1 - |w|^2)f'(w) = (f \circ \varphi_w)'(0) \) and \( (1 - |z|^2)(f \circ \varphi_w)'(z) = (1 - |\lambda|^2)(f')(\lambda) \).

Let \( g = f \circ \varphi_w \), then \( g'(0) = (1 - |w|^2)f'(w) \) and \( (1 - |\lambda|^2)g'(\lambda) = (1 - |z|^2)(f \circ \varphi_w)'(z) \). Thus

\[
|1 - |z|^2||f'(z)| - (1 - |w|^2)|f'(w)|| = \left| (1 - |\lambda|^2)g'(\lambda) - g'(0) \right|
\]
\[
= |\lambda|^2|g'(0)| + (1 - |\lambda|^2)|g'(\lambda) - g'(0)|
\]

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By definition of \( \|g\|_B \),
\[
|g'(0)| \leq \|g\|_B = \|f \circ \varphi_w\|_B = \|f\|_B.
\]
Now estimate \(|g'(\lambda) - g'(0)|\). Let \( \lambda \in \mathbb{D} \). For each \( z \in \mathbb{D} \),
\[
(1 - |z|^2)g''(z) = (g' \circ \varphi_z)'(0) = \frac{1}{2\pi i} \int_0^{2\pi} g' \circ \varphi_z(re^{i\theta})e^{-i\theta}d\theta
\]
For each fixed \( 0 < r < 1 \),
\[
\left| \int_0^{2\pi} g' \circ \varphi_z(re^{i\theta})e^{-i\theta}d\theta \right| \leq \int_0^{2\pi} (1 - |\varphi_z(re^{i\theta})|^2) \frac{g' \circ \varphi_z(re^{i\theta})}{1 - |\varphi_z(re^{i\theta})|^2}d\theta
\]
Also,
\[
\frac{1}{1 - |\varphi_z(re^{i\theta})|^2} = \frac{|1 - re^{i\theta}\bar{z}|^2}{(1 - |z|^2)(1 - r^2)}
\]
Thus for \( 0 < r < 1 \),
\[
(1 - |z|^2)^2 |g''(z)| \leq \|g\|_B \frac{1 + r^2|z|^2}{r(1 - r^2)}.
\]
Case (i) If \( |\lambda| \leq \frac{1}{2} \), let \( h(r) = \frac{1 + r^2}{r(1 - r^2)} \). The minimum value of \( h(r) \) for \( 0 < r < 1 \) is roughly 2.81. Hence for \( |z| \leq \frac{1}{2} \), \( (1 - |z|^2)^2 |g''(z)| \leq 2.81\|g\|_B \). Also
\[
|g'(\lambda) - g'(0)| \leq \int_0^1 |g''(t\lambda)||\lambda|dt.
\]
Hence
\[
|g'(\lambda) - g'(0)| \leq 2.81\|g\|_B \int_0^1 \frac{|\lambda|}{(1 - t^2|\lambda|^2)^2} = \int_0^{|\lambda|} \frac{ds}{(1 - s^2)^2}
\]
and
\[
\int_0^{|\lambda|} \frac{ds}{(1 - s^2)^2} = \frac{1}{4} \left( \frac{2|\lambda|}{1 - |\lambda|^2} + \log \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right) \right).
\]
So
\[
(1 - |\lambda|^2)|g'(\lambda) - g'(0)| \leq 2.81\|g\|_B |\lambda| = 2.81\|f\|_B |\lambda|.
\]
Thus if \( |\lambda| \leq \frac{1}{2} \),
\[
\left| (1 - |\lambda|^2)|g'(\lambda)| - |g'(0)| \right| \leq |\lambda|^2\|f\|_B + 2.81|\lambda|\|f\|_B \leq 3.31\|f\|_B.
\]
Case (ii) If $|\lambda| > \frac{1}{2}$, $2|\lambda| > 1$. Then
\[
\left| (1 - |\lambda|^2)g'(\lambda) - |g'(0)| \right| \leq \max\{(1 - |\lambda|^2)|g'(\lambda)|, |g'(0)|\}
\leq ||g||_B = ||f||_B \leq 2|\lambda|||f||_B.
\]
Combining both cases,
\[
\left| (1 - |\lambda|^2)g'(\lambda) - |g'(0)| \right| \leq 3.31|\lambda|||f||_B
\]
for $|\lambda| < 1$. Thus
\[
\left| (1 - |z|^2)|f'(z)| - (1 - |w|^2)|f'(w)| \right| \leq 3.31|\lambda|||f||_B = 3.31\tilde{p}(z, w)||f||_B
\]
\[
\square
\]

4.2 Compact Composition Operators on $B$

If $X$, $Y$ are Banach spaces, $T : X \to Y$ a bounded linear operator, $T$ is called weakly compact if $T$ takes bounded sets in $X$ to relatively weakly compact sets in $Y$. Gantmacher’s theorem states that $T$ is weakly compact if and only if $T^{**}(X^{**}) \subset Y^{**}$, where $T^{**}$ is the second adjoint of $T$.

Lemma 4.6. $K \subset B_0$, for $K$ a closed set, is compact if and only if $K$ is a bounded set and
\[
\lim_{|z| \to 1} \sup_{f \in K} (1 - |z|^2)|f'(z)| = 0
\]
(15)

Proof. $\Rightarrow$ Let $\epsilon > 0$ and choose an $\frac{\epsilon}{2}$-net $f_1, \ldots, f_n$ in $K$. $\exists r$ with $0 < r < 1$ such that $(1 - |z|^2)|f'_i(z)| < \frac{\epsilon}{2}$ if $|z| < r$, $\forall i$.

If $f \in K$, then $||f - f_i||_B < \frac{\epsilon}{2}$ for some $f_i$ and so
\[
(1 - |z|^2)|f'(z)| \leq ||f - f_i||_B + (1 - |z|^2)|f'_i(z)| < \epsilon, \text{ when } |z| > r.
\]

$\Leftarrow$ If $K$ closed bounded set satisfying the condition (15) and $(f_n)$ a sequence in $K$, then Montel’s theorem implies that $\exists$ subsequence $(f_{n_k})$ which converges uniformly on compact subsets of $D$ to some holomorphic $f$.

Then $f'_{n_k} \to f'$ uniformly on compact subsets of $D$. If $\epsilon > 0$, $\exists r < 1$ such that $\forall g \in K$, $(1 - |z|^2)|g'(z)| < \frac{\epsilon}{2}$ if $|z| > r$. Thus $(1 - |z|^2)|f'(z)| < \frac{\epsilon}{2}$ if $|z| > r$.

Since $f_{n_k} \to f$ uniformly and $f'_{n_k} \to f'$ uniformly on $|z| \leq r$, $\limsup_{k \to \infty} ||f - f_{n_k}||_B \leq \epsilon$.

Since $\epsilon > 0$, $\lim_{k \to \infty} ||f - f_{n_k}||_B = 0 \Rightarrow K$ is compact. \[
\square
\]

Theorem 4.7. If $\varphi : D \to D$ holomorphic, then $\varphi$ induces a compact composition operator on $B_0$ if and only if
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)}|\varphi'(z)| = 0.
\]
(16)
Proof. Lemma 4.6 implies that \( C_\varphi \) compact on \( \mathcal{B}_0 \) if and only if

\[
\lim_{|z| \to 1} \sup_{\|f\| \leq 1} (1 - |z|^2)(f \circ \varphi)'(z) = 0.
\]

But

\[
(1 - |z|^2)(f \circ \varphi)'(z) = \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)|(1 - |\varphi(z)|^2)|f'(\varphi(z))|
\]

and

\[
\sup_{\|f\| \leq 1} (1 - |w|^2)|f'(w)| = 1
\]

for \( w \in \mathbb{D} \).

Note that condition (16) implies \( \varphi \in \mathcal{B}_0 \).

Theorem 4.8. If \( \varphi : \mathbb{D} \to \mathbb{D} \) holomorphic, then \( \varphi \) induces a compact composition operator on \( \mathcal{B} \) if and only if, for every \( \epsilon > 0 \), \( \exists 0 < r < 1 \) such that

\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)}|\varphi'(z)| < \epsilon
\]

when \( |\varphi(z)| > r \).

Proof. \( \Leftarrow \) Enough to show that if \( (f_n) \) is a bounded sequence in \( \mathcal{B} \) converging uniformly to 0 on compact subsets of \( \mathbb{D} \), then \( \|f_n \circ \varphi\| \to 0 \).

Let \( M = \sup_n \|f_n\| \). Given \( \epsilon > 0 \), \( \exists 0 < r < 1 \) such that

\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)}|\varphi'(z)| < \frac{\epsilon}{2M}
\]

if \( |\varphi(z)| > r \). Since

\[
(1 - |z|^2)((f_n \circ \varphi)'(z)) = \frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)}|\varphi'(z)|(1 - |\varphi(z)|^2)|f_n'(\varphi(z))|
\]

\[
\leq M \frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)}|\varphi'(z)|
\]

then

\[
(1 - |z|^2)((f_n \circ \varphi)'(z)) < \frac{\epsilon}{2}
\]

if \( |\varphi(z)| > r \). But \( f_n \circ \varphi(0) \to 0 \) and \( (1 - |w|^2)|f_n'(w)| \to 0 \) uniformly for \( |w| \leq r \).

Since

\[
(1 - |z|^2)((f_n \circ \varphi)'(z)) \leq (1 - |\varphi(z)|^2)|f_n'(\varphi(z))|
\]

for large enough \( n \), then \( |f_n \circ \varphi(0)| < \frac{\epsilon}{2} \) and \( (1 - |z|^2)((f_n \circ \varphi)'(z)) < \frac{\epsilon}{2} \) if \( |\varphi(z)| > r \).

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Hence \( \|f_n \circ \varphi\|_B < \epsilon \) for large enough \( n \).

\[ \Rightarrow \) Assume (17) fails. Then \( \exists \) subsequence \((z_n)\) in \( \mathbb{D} \) and \( \epsilon > 0 \) such that 
\[ |z_n| \to 1 \text{ and } \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)} |\varphi'(z)| > \epsilon, \quad \forall n \]
Passing to a subsequence if necessary, without loss of generality, \( w_n = \varphi(z_n) \to w_0 \in \mathbb{T} \). Let
\[ f_n(z) = \log \left( \frac{1}{1-w_n z} \right) \]
\( f_n \to f_0 \) uniformly on compact subsets of \( \mathbb{D} \). But
\[ \|C_{\varphi} f_n - C_{\varphi} f_0\|_B \geq (1 - |z_n|^2)(|C_{\varphi} f_n)'(z_n) - (C_{\varphi} f_0)'(z_n)| \]
\[ = (1 - |z_n|^2)|\varphi'(z_n)| \left| \frac{w_n - w_0}{1 - w_n w_0} \right| > \epsilon, \quad \forall n. \]
So \( C_{\varphi} f_n \) does not converge to \( C_{\varphi} f_0 \) in norm. Thus \( C_{\varphi} \) not compact. \( \square \)

(16) \( \Rightarrow \) (17), since in this case, \( C_{\varphi} \) on \( \mathcal{B} \) is the second adjoint of \( C_{\varphi} \) of \( \mathcal{B}_0 \), but the two conditions are not equivalent. (16) \( \Rightarrow \varphi \in \mathcal{B}_0 \), while \( \exists \varphi \notin \mathcal{B}_0 \) satisfying (17) eg. any \( \varphi \) such that \( \|\varphi\|_\infty < 1 \) satisfies (17).

A sequence \((w_n) \subset \mathbb{D}\) is \( \eta \)-separated if
\[ \tilde{\rho}(w_n, w_m) = \left| \frac{w_m - w_n}{1 - w_m \bar{w}_n} \right| > \eta, \text{ when } m \neq n. \]
Any sequence \((w_n)\) such that \( |w_n| \to 1 \) possesses an \( \eta \)-separated subsequence for any \( \eta > 0 \).

**Proposition 4.9.** There is an absolute constant \( R > 0 \) such that if \((w_n)\) is \( R \)-separated, then for every bounded sequence \( \lambda_n, \exists f \in \mathcal{B} \) such that
\[ (1 - |w_n|^2)f'(w_n) = \lambda_n, \quad \forall n. \]

**Proof.** Proof is omitted. See [7]. \( \square \)

**Theorem 4.10.** Every weakly compact composition operator \( C_{\varphi} \) on \( \mathcal{B}_0 \) is compact.

**Proof.** \( C_{\varphi} : \mathcal{B}_0 \to \mathcal{B}_0 \) is compact if and only if
\[ \lim_{|z| \to 1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)} |\varphi'(z)| = 0 \]
and (Gantmacher) weakly compact if and only if \( C_{\varphi} f \in \mathcal{B}_0 \forall f \in \mathcal{B}_0 \).

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If $C\varphi$ not compact, $\exists \epsilon > 0$ and a sequence $(z_n)$ with $|z_n| \to 1$ such that
\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| \geq \epsilon, \quad \forall n.
\]
Since $\varphi \in B_0$, $|\varphi(z_n)| \to 1$, and, passing to a subsequence, $(\varphi(z_n))$ is $R$-separated. Proposition 4.9 implies that $\exists f \in B$ such that
\[
(1 - |\varphi(z_n)|^2) |(C\varphi f)'(z_n)| = 1, \quad \forall n.
\]
Since $(1 - |z|^2)|C\varphi f)'(z_n)| \geq \epsilon$ and $|z_n| \to 1$, then $C\varphi f \not\in B_0$, so $C\varphi$ is not weakly compact.

\varphi has angular derivative at $\xi \in \mathbb{T}$ if the nontangential limit $w = f(\xi) \in \mathbb{T}$ exists and if $\frac{f(z) - f(\xi)}{z - \xi}$ converges to some $\mu \in \mathbb{C}$ as $z \to \xi$ nontangentially.

If $\varphi$ has finite angular derivative at some point of $\mathbb{T}$, $C\varphi$ cannot be compact. $\mu \neq 0$ and the Julia-Caratheodory Lemma (see [1, p.82]) gives
\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| \to \xi \mu \neq 0
\]
with the convergence being nontangential. Then Theorems 4.7 or 4.8 give that $C\varphi$ not compact.

If $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, $\varphi \in B_0^h$, the hyperbolic little Bloch class, if
\[
\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty, \text{ and } \lim_{|z| \to 1} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0.
\]
Note $B_0^h$ is not a linear space since $\varphi$ is required to map $\mathbb{D}$ into itself. Thus $C\varphi : B_0 \to B_0$ compact if and only if $\varphi \in B_0^h$.

EXAMPLE 3) $\varphi$ can push $\mathbb{D}$ much more sharply into itself and still induce a non-compact composition operator. Consider the functions
\[
\varphi_{\lambda,\alpha}(z) = 1 - \lambda(1 - z)^\alpha, \quad 0 < \lambda, \alpha < 1.
\]
$\varphi_{\lambda,\alpha} \in B_0$ and maps $\mathbb{D}$ onto a region which behaves like a Stolz angle of opening $\pi \alpha$. If $C\varphi$ were compact on $B_0$, composition with $\log \left(\frac{1}{1 - z}\right)$ would yield a function in $B_0$, but this is not so, since if
\[
g(z) = \log \left(\frac{1}{1 - \varphi_{\lambda,\alpha}(z)}\right)
\]
then
\[
(1 - |z|^2)|g'(z)| = (1 - |z|^2) \left| \frac{\alpha}{1 - z} \right| \to 0
\]
as \(|z| \to 1\).

**EXAMPLE 4) Composition Operators on Bergman and Hardy Spaces.**

For \(p > 0\), the Hardy space \(H^p\) is the space of all analytic functions on \(D\) such that
\[
||f||_{H^p} = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.
\]
For \(p > 0\), the Bergman space \(A^p\) is the space of analytic functions on \(D\) such that
\[
||f||_{A^p} = \int_D |f|^p dA < \infty
\]
where \(A\) is Lebesgue measure on \(D\). The following theorem, ([8] Theorem 6.4), shows that if \(\varphi \in B_h^0\), then \(C_\varphi : A^p \to H^q\) is bounded.

**Theorem 4.11.** If \(\varphi : \mathbb{D} \to G \subset \mathbb{D}\), with \(G\) simply connected and
\[
\lim_{|w| \to 1} \frac{d_\varphi(w)}{1 - |w|} = 0
\]
then \(C_\varphi : A^p \to H^q\) is bounded \(\forall 0 < p \leq q < \infty\).

Note that (18) is equivalent to \(\varphi \in B_h^0\), because the Koebe one quarter theorem implies \(d_\varphi(w)\) is comparable with \((1 - |w|^2)|\varphi'(w)|\).

**Proof.** Without loss of generality, \(0 \in G\) and \(\varphi(0) = 0\). By [8] Theorem 1.1b, \(C_\varphi : A^p \to H^q\) bounded if and only if \(1 - |z| = O((1 - |\varphi(z)|)^p)\) as \(|z| \to 1\) and for all \(\eta > 0\). Taking logarithms, and setting \(\varphi(z) = w\), it remains to show that for a given \(\eta > 0\), there is a \(C\) such that
\[
\eta \rho_D(0, w) \leq C + \rho_G(0, w), \quad w \in G.
\]
Choose \(r \in (0, 1)\) such that \(4\eta d_\varphi(z) \leq 1 - |z|\) if \(|z| > r\) and \(z \in G\). Set \(C = \eta \rho_D(0, r)\). The condition above is clearly satisfied for \(|w| \leq r\), so assume \(|w| > r\). Now, the quasi-hyperbolic metric is defined by
\[
k_G(z_1, z_2) = \left\{ \int_{\gamma} \frac{ds}{d_\varphi(z)} : \gamma \text{ is an arc in } G \text{ from } z_1 \text{ to } z_2 \right\}
\]
and satisfies \(\rho_G \leq 2k_G \leq 4\rho_G\). So
\[
\eta \rho_D(0, w) = C + \eta \rho_D(r, |w|) \leq C + 2\eta k_G(r, |w|) = C + 2\eta \int_r^{|w|} \frac{dt}{1 - t}
\]
For \(\gamma\) an arc in \(G\) from 0 to \(w\),
\[
\int_r^{|w|} \frac{dt}{1 - t} \leq \int_{\gamma \cap (\mathbb{D} \cup \overline{\mathbb{D}})} \frac{ds}{4\rho_G(z)} \leq \int_{\gamma} \frac{ds}{4\rho_G(z)}
\]
Take the infimum over all such arcs \(\gamma\) and then
\[
\eta \rho_D(0, w) \leq C + 2\eta \int_r^{|w|} \frac{dt}{1 - t} \leq C + 0.5k_G(0, w) \leq C + \rho_G(0, w)
\]

\(\square\)

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This then implies that $C_\varphi : A^p \to H^q$ is compact if $\varphi \in B_0^p$.

### 4.3 Cusps

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be univalent, $G = \varphi(\mathbb{D})$ and assume $\overline{G} \cap \mathbb{T} = \{1\}$. $G$ has a cusp at 1 if

$$\text{dist}(w, \partial G) = o(|1 - w|) \text{ as } w \to 1 \text{ in } G. \quad (19)$$

Otherwise $G$ does not have a cusp at 1. The cusp is nontangential if $G$ lies in a Stolz angle at 1 i.e. $\exists \ r, M > 0$ such that $|1 - w| \leq M (1 - |w|^2)$ if $|1 - w| < r$, and $w \in G$.

(19) can be reformulated. Let $w = \varphi(z)$ and let $z \to \xi$ as $w \to 1$. Since the Koebe one quarter theorem says that $\text{dist}(\varphi(z), \partial G)$ is comparable to $(1 - |z|^2)|\varphi'(z)|$, then $G$ has a cusp at 1 if $(1 - |z|^2)|\varphi'(z)| = o(|1 - \varphi(z)|)$ as $z \to \xi$. So

$$\frac{(1 - |z|^2)|\varphi'(z)|}{|1 - \varphi(z)|} \to 0, \text{ as } z \to \xi$$

**Theorem 4.12.** With notation as above, if $G$ does not have a cusp at 1, then $C_\varphi$ is not compact on $B_0^p$.

**Proof.** Suppose (19) fails. Then $\exists \delta > 0$ and $(z_n) \subset \mathbb{D}$ such that $|z_n| \to 1$ but

$$\text{dist}(\varphi(z_n), \partial G) \geq \delta |1 - \varphi(z_n)|.$$ 

Hence

$$\delta (1 - |\varphi(z_n)|^2) \leq 2\delta (1 - |\varphi(z_n)|) \leq 2 \text{dist}(\varphi(z_n), \partial G) \leq 2 (1 - |z_n|^2)|\varphi'(z_n)|$$

So

$$\frac{(1 - |z_n|^2)}{(1 - |\varphi(z_n)|^2)} |\varphi'(z_n)| \geq \frac{\delta}{2}$$

Since $|z_n| \to 1$, Theorem 4.7 $\Rightarrow C_\varphi$ not compact.

**Theorem 4.13.** If $\varphi$ univalent and $G$ has a nontangential cusp at 1 and touches $\mathbb{T}$ at no other point, then $C_\varphi$ is compact on $B_0^p$.

**Proof.** $\varphi \in B_0^p$, so enough to show

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| = 0$$

then Theorem 4.7 applies. $\exists M, r$ such that $|1 - w| \leq M (1 - |w|^2)$ if $|1 - w| < r$ and $w \in G$.

Let $\epsilon > 0$. Since $G$ has a cusp at 1, $\exists \delta > 0$ such that

$$\text{dist}(w, \partial G) \leq \frac{\epsilon}{4M} |1 - w|$$
if \( |1 - w| < \delta \) and \( w \in G \). Let \( \eta = \min(\delta, r) \). If \( |1 - \varphi(z)| < \eta \),
\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| \leq \frac{4 \text{dist}(\varphi(z), \partial G)}{1 - |\varphi(z)|^2}
\]
\[
\leq \frac{\epsilon |1 - \varphi(z)|}{M (1 - |\varphi(z)|^2)} < \epsilon.
\]
But if \( |1 - \varphi(z)| \geq \eta \), \( \exists \) constant \( N > 0 \) such that \( |\varphi'(z)| \leq N \) by smoothness assumption of \( \varphi \), and there is a \( \alpha > 0 \) such that \( 1 - |\varphi(z)|^2 \geq \alpha \). Then
\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| \leq \frac{N}{\alpha} (1 - |z|^2) < \epsilon
\]
if \( |z|^2 > 1 - \frac{\alpha \epsilon}{N} \).

**EXAMPLE 5**) Can find regions \( G \) with tangential cusp such that \( \varphi : \mathbb{D} \to G \subset \mathbb{D} \) allows \( C_\varphi \) to be compact or non-compact.

Let \( h(\theta), k(\theta) \) be positive continuous functions on \( [0, \theta_0] \) with \( h(\theta) = o(\theta) \) and \( k(\theta) = o(\theta) \). Let
\[
G = \{ re^{i\theta} : 0 < \theta < \theta_0, \ h(\theta) < 1 - r < h(\theta) + k(\theta) \}
\]
Then \( G \) has tangential cusp at 1.
(i) If \( k(\theta) = o(h(\theta)) \), then for \( w = re^{i\theta} = \varphi(z) \),
\[
(1 - |z|^2) |\varphi'(z)| \leq \text{dist}(w, \partial G) \leq k(\theta)
\]
and \( (1 - |w|^2) \geq 1 - |w| > h(\theta) \). So
\[
\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)} |\varphi'(z)| \to 0
\]
as \( |\varphi(z)| \to 1 \). \( \varphi \) univalent, so Theorem 4.13 gives that \( C_\varphi \) is compact.

(ii) If \( k(\theta) = 2h(\theta) \) and \( w(\theta) = (1 - 2h(\theta))e^{i\theta} = \varphi(z(\theta)) \) then \( \text{dist}(w(\theta), \partial G) > ch(\theta) \) for constant \( c \). Since \( (1 - |z|^2) |\varphi'(z)| \geq \text{dist}(\varphi(z), \partial G) \),
\[
\frac{1 - |z(\theta)|^2}{(1 - |w(\theta)|^2)} |\varphi'(z(\theta))| \geq \frac{c}{4}
\]
so \( C_\varphi \) is not compact.

## 5 Inner functions and \( B \)

### 5.1 Inner functions

Let \( \{ z_n \} \subset \mathbb{D} \) be such that \( \sum (1 - |z_n|) < \infty \), and let \( m \) be the number of the \( z_n \) which are zero, then the **Blaschke product**
\[
B(z) = z^m \prod_{|z_n| \neq 0} \frac{-z_n}{z_n} \frac{z - z_n}{|z_n| (1 - \frac{z_n}{z})}
\]
converges on $D$ and is analytic. $\|B\|_\infty \leq 1$ and the zeros of $B(z)$ are the points $z_n$ with multiplicity corresponding to the number of times it occurs in the sequence $\{z_n\}$. $|B(e^{i\theta})| = 1$ almost everywhere. The purpose of the convergence factors $\frac{1}{|z_n|}$ is to ensure that the argument of the infinite product converges.

A sequence $\{z_j\} \subset D$ is called an interpolating sequence if every interpolation problem $f(z_j) = a_j$ for $j = 1, 2, ...$ with $\{a_j\}$ bounded has a solution $f \in H^\infty$. A Blaschke product is called interpolating if it has distinct zeros and the zeros form an interpolating sequence.

A singular inner function is of the form $S(z) = \exp \left( - \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \right)$ where the measure $d\mu$ is positive and singular to $d\theta$.

Also note that an outer function, for $F \in H^\infty$ is of the form $G(z) = \exp \left( \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |F(e^{i\theta})| \frac{d\theta}{2\pi} \right)$.

An inner function is a function $f(z) \in H^\infty = \{f : f$ bounded analytic in $D\}$ such that $|f(e^{i\theta})| = 1$ almost everywhere. For example, every Blaschke product is inner. Every inner function factors as $f(z) = e^{ic} B(z) S(z)$ (20) where $c \in \mathbb{R}$, $B$ a Blaschke product, and $S$ a singular function. Also, every $F \in H^\infty$ factors as $F(z) = C \ast B(z) G(z) S(z)$ where $C$ is a constant and $G$ is an outer function.

A compact set $K \subset \mathbb{C}$ has positive logarithmic capacity if there is a positive measure $\sigma$ on $K$ with $\sigma \neq 0$ such that the logarithmic potential

$$U_\sigma(z) = \int_K \log \left( \frac{1}{|\xi - z|} \right) \, d\sigma(\xi)$$

is bounded on some neighbourhood of $K$.

**Theorem 5.1 (Frostman).** Let $f(z)$ be a non-constant inner function on $D$. Then $\forall \xi$ with $|\xi| < 1$, except on a set of capacity 0, the function

$$f_\xi(z) = \frac{f(z) - \xi}{1 - \bar{\xi} f(z)}$$

is a Blaschke product.

**Proof.** Proof omitted. See [5].
Corollary 5.2. The set of Blaschke products is uniformly dense in the set of inner functions.

Proof. Follows from Theorem 5.1.

Let \( f \in H^\infty(\mathbb{D}) \) and \( z_0 \in \mathbb{T} \). The cluster set of \( f \) at \( z_0 \) is

\[
\text{Cl}(f, z_0) = \bigcap_{r > 0} f(\mathbb{D} \cap \mathbb{D}(z_0, r))
\]

\( \xi \in \text{Cl}(f, z_0) \) if and only if \( \exists \) points \( z_n \in \mathbb{D} \) tending to \( z_0 \) such that \( f(z_n) \to \xi \). The cluster set is a compact non-empty connected plane set, and is a singleton if and only if \( f \) is continuous on \( \mathbb{D} \cup \{z_0\} \).

The range set of \( f \) at \( z_0 \) is

\[
\text{R}(f, z_0) = \bigcap_{r > 0} f(\mathbb{D} \cap \mathbb{D}(z_0, r))
\]

so \( \xi \in \text{R}(f, z_0) \) if and only if \( \exists \) points \( z_n \in \mathbb{D} \) tending to \( z_0 \) such that \( f(z_n) = \xi \) for \( n = 1, 2, \ldots \), i.e. the range set is the set of values assumed infinitely often in each neighbourhood of \( z_0 \). Clearly \( \text{R}(f, z_0) \subset \text{Cl}(f, z_0) \).

If \( f(z) \) analytic across \( z_0 \) and non-constant, then \( \text{Cl}(f, z_0) = f(z_0) \) and \( \text{R}(f, z_0) = \phi \).

Theorem 5.3. Let \( f(z) \) be an inner function on \( \mathbb{D} \) and \( z_0 \in \mathbb{T} \) a singularity of \( f(z) \). Then \( \text{Cl}(f, z_0) = \mathbb{D} \) and \( \text{R}(f, z_0) = \mathbb{D} \setminus L \), where \( L \) is a set of logarithmic capacity zero.

Proof. Proof omitted. See [5].

Thus the boundary behaviour of a \( H^\infty \) function can be wild. Eg. If \( f(z) \) is a Blaschke product whose zeros are dense on \( \mathbb{T} \), then the theorem holds at every \( z_0 \in \mathbb{T} \), even though \( f(z) \) has nontangential limits almost everywhere.

5.2 Inner functions in \( \mathcal{B} \) and \( \mathcal{B}_0 \)

Proposition 3.8 implies that there are singular inner functions contained in \( \mathcal{B} \), and it is also clear that the finite Blaschke products are contained in \( \mathcal{B} \), but it is less clear that infinite Blaschke products may be in \( \mathcal{B} \).

Lemma 5.4. Let \( \Omega \subset \mathbb{D} \) and \( f \) analytic in \( \mathbb{D} \) with \( f(\mathbb{D}) \subset \Omega \). Then \( \forall z \in \mathbb{D}, \)

\[
(1 - |z|^2)|f'(z)| \leq 6 \text{dist}(f(z), \partial \Omega) \log \left( \frac{e}{\text{dist}(f(z), \partial \Omega)} \right).
\]

Proof. Let \( a \in \Omega \) be such that \( \text{dist}(f(z), \partial \Omega) = |f(z) - a| \) and assume for now \( |f(z) - z| \geq \frac{1}{4}(1 - |f(z)|^2) \). Then

\[
(1 - |z|^2)|f'(z)| \leq 1 - |f(z)|^2 \leq 4|f(z) - a| \leq 6|f(z) - a| \log \left( \frac{e}{|f(z) - a|} \right)
\]

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If now
\[ |f(z) - a| < \frac{1}{4} (1 - |f(z)|^2), \] (21)
then \( a \in \mathbb{D} \), i.e. \( a \in \mathbb{T} \). Since
\[ \pi(z) = \exp \left( \frac{1 + z}{1 - z} \right) \]
is the universal covering map of \( \mathbb{D} \setminus \{0\} \), there is a holomorphic \( \phi: \mathbb{D} \to \mathbb{D} \) such that
\[ \frac{f - a}{1 - \pi f} = \pi \circ \phi. \]
For \( w \in \mathbb{D} \),
\[ (1 - |w|^2)|\pi'(w)| = 2|\pi(w)|\log(|\pi(w)|^{-1}). \]
Thus
\[ \left( \frac{1 - |z|^2}{1 - |\pi f(z)|^2} \right) |f'(z)| \leq (1 - |\varphi(z)|^2)|\pi'(\varphi(z))| \]
\[ = 2 \left| \frac{f(z) - a}{1 - \pi f(z)} \right| \log \left| \frac{f(z) - a}{1 - \pi f(z)} \right|^{-1}. \]
Thus
\[ (1 - |z|^2)|f'(z)| \leq 2 \frac{|1 - \pi f(z)|}{1 - |a|^2} |f(z) - a| \log \left( \frac{e}{|f(z) - a|} \right) \]
Result follows from (21).

**Lemma 5.5.** Let \( h: (0, 1] \to (0, 1] \) be continuous. \( \exists \) a countable set \( \Lambda \subset \mathbb{D} \setminus \{0\} \) whose cluster set is contained in \( \mathbb{T} \) such that \( \forall z \in \mathbb{D}, \text{dist}(z, \Lambda \cup \mathbb{T}) \leq h(1 - |z|) \).

**Proof.** Proof omitted. \( \square \)

**Theorem 5.6.** Let \( \varphi: (0, 1] \to [0, \infty) \) be a continuous function and
\[ \lim_{z \to 0} \varphi(0) = 0. \]
Then \( \exists \) interpolating Blaschke product \( B \) such that
\[ (1 - |z|^2)|B'(z)| \leq \varphi(1 - |B(z)|^2) \]
\( \forall z \in \mathbb{D} \).

**Proof.** Given \( \varphi(t) \), consider continuous \( h: (0, 1] \to (0, 1] \) such that
\[ 6h(t) \log \left( \frac{e}{h(t)} \right) \leq \varphi(t), \quad \forall t \in (0, 1]. \]
For the set \( \Lambda \) of Lemma 5.5, let \( B \) be the universal covering of \( \mathbb{D} \) onto \( \Omega = \mathbb{D} \setminus \Lambda \). Lemmas 5.4, 5.5 give
\[ (1 - |z|^2)|B'(z)| \leq \varphi(1 - |B(z)|^2) \]
as required. So required to prove that $B$ is an interpolating Blaschke product.

Since $B \in H^\infty$, its radial limit $B(\xi)$ exists for almost every $\xi \in \mathbb{T}$. Moreover, $B$ is a covering map, so $B(\xi) \in \Lambda \cup \mathbb{T}$. Hence $B(\xi) \in \mathbb{T}$ for almost every $\xi \in \mathbb{T}$ since $\Lambda$ is countable. Thus $B$ is inner. If $B$ had a singular inner factor, then there would be at least one value of $\xi \in \mathbb{T}$, $\xi_0$ say, with

$$\lim_{r \uparrow 1} B(r \xi_0) = 0.$$ 

$0 \notin \Lambda$ so this cannot happen. (20) gives that $B$ is a Blaschke product. To prove it is interpolating, enough to observe that

$$(1 - |z|^2)|B'(z)|$$

depends only on $B(z)$. If $B(a) = B(b)$, there is an automorphism $\phi$ of $\mathbb{D}$ such that $\phi(a) = b$ and $B \circ \phi = B$. Hence,

$$(1 - |b|^2)|B'(b)| = (1 - |a|^2)|\phi'(a)||B'(b)| = (1 - |a|^2)|B'(a)|.$$

Thus

$$\inf_n \{(1 - |z_n|^2)|B'(z_n)| : B(z_n) = 0\} \geq \delta > 0$$

for some $\delta$ and this characterises interpolating Blaschke products. \hfill \Box

So by taking $\varphi(t) = t$, Theorem 5.6 gives that there are infinite Blaschke products in $\mathcal{B}$. Theorem 5.6 gives the following corollary, showing that the interpolating Blaschke product constructed in the proof gives a compact composition operator on $\mathcal{B}$.

**Corollary 5.7.** \exists interpolating Blaschke product $B$ such that $C_B : \mathcal{B} \to \mathcal{B}$ is compact.

**Proof.** From Theorems 4.8 and 5.6. \hfill \Box

If $w : [0, 1) \to \mathbb{R}$ is a positive continuous function such that $\lim_{t \uparrow 1} w(t) = \infty$, then

$$H(w) = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ with norm } ||f||_w = \sup_{z \in \mathbb{D}} \left\{ \frac{|f(z)|}{w(|z|)} \right\} < \infty \right\}$$

**Corollary 5.8.** For any such $w$ and $\epsilon > 0$, \exists interpolating Blaschke product $B$ such that composition operator $C_B$ maps $H(w)$ into $\mathcal{B}$ and $||C_B f||_B \leq \epsilon ||f||_w$ for any $f \in H(w)$.

**Proof.** Without loss of generality, $\epsilon = 1$, otherwise replace $w$ by $\epsilon^{-1} w$. If $f \in H(w)$ and $||f||_w = 1$, then by the Cauchy inequality,

$$(1 - |z|^2)|f'(z)| \leq 4w(|z| + \frac{1}{2}(1 - |z|))$$

Choose $\varphi(t)$ so that

$$w(t + \frac{1}{2}(1 - t))\varphi(1 - t^2) \leq 1$$

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for $0 \leq t < 1$. Then $\varphi(t) \to 0$ as $t \to 0$. Theorem 5.6 $\Rightarrow \exists$ interpolating Blaschke product $B$ such that
\[ (1 - |z|^2)|B'(z)| \leq \varphi(1 - |B(z)|^2), \quad z \in \mathbb{D} \]
Hence $\forall z \in \mathbb{D},$
\[ (1 - |z|^2)|(f \circ B)'(z)| \leq 1. \]

In the above proof, could have chosen $\varphi(t)$ so that $w(t + \frac{1}{2}(1-t))\varphi(1-t^2) \to 0$ as $t \uparrow 1$. Then using Corollary 5.7, could have $C_B : H(w) \to B$ compact.

Moving on to the question of whether there are inner functions in $B_0$, it is clear again that finite Blaschke products are contained in $B_0$, but it is far from clear there are any infinite Blaschke products. K. Stephenson in [10] constructed an inner function in $B_0$ by building a Riemann image surface via “cutting and pasting” techniques and checking that the resulting function was inner and in $B_0$.

C. Bishop in [11] gives a construction of an infinite Blaschke product, which will be briefly outlined here. Let $I \subset \mathbb{T}$ be a segment and define the Carleson square with base $I$
\[ Q(I) = \{ re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r \leq 1 \}. \]

Let $l(Q) = |I|$ be the side length and for constant $N$, let $NQ$ be the Carleson square with $l(NQ) = Nl(Q)$. Let $P_Q$ be the Poisson kernel for $z$ at the centre of the top edge of $Q$. Recall that for $F \in H^\infty$, $F$ factors as $F(z) = C \ast B(z)G(z)S(z)$ where
\[ B(z) = z^m \prod_{|z_n| \neq 0} \frac{z - z_n}{|z_n| (1 - \bar{z_n}z)} \]
\[ S(z) = \exp \left( - \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\lambda(e^{i\theta}) \right) \]
\[ G(z) = \exp \left( \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |F(e^{i\theta})| \frac{d\theta}{2\pi} \right). \]

So associated to a bounded function $F$, there is a measure $\mu$ on $\mathbb{D}$ given by
\[ d\mu = \Sigma\delta_{z_n}(1 - |z_n|) + d\lambda - \log|F|\frac{d\theta}{2\pi} = d\nu + d\sigma \]
where $\nu$ is the positive measure $\nu = \Sigma\delta_{z_n}(1 - |z_n|)$ and $\delta_{z_n}$ is the point mass at $z_n$. $\nu$ is supported on $\mathbb{D}$ and $\sigma$ is supported on $\mathbb{T}$.

Let $Q' \subset Q$ be the Carleson square of half the size. Then Bishop proved
\textbf{Theorem 5.9.} Let $F$ be in the unit ball of $H^\infty(\mathbb{D})$. Then $F \in \mathcal{B}_0$ if and only if for every $\epsilon > 0$, $\exists N, \delta > 0$ such that $l(Q) < \delta$ implies one of the following two conditions hold

\begin{enumerate}[(i)]
  \item $\frac{\mu(Q)}{l(Q)} > \frac{1}{\epsilon}$
  \item $\left| \frac{\mu(Q)}{l(Q)} - \frac{\mu(Q')}{l(Q')} \right| < \epsilon$ and $\int_{(NQ)^c} P_Q(w)d\mu(w) < \epsilon$.
\end{enumerate}

[11] also gives an extended example on how to explicitly construct an infinite Blaschke product in $\mathcal{B}_0$.

\section{Inner Functions in $\mathcal{B}_0^h$}

Recall $\varphi \in \mathcal{B}_0^h$ if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty \quad \text{and} \quad \lim_{|z| \to 1} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0.$$ 

Such a function cannot extend analytically to any point of $T$. If $\varphi$ has an angular derivative at $\xi \in T$, then the Julia-Caratheodory Lemma asserts

$$\liminf_{z \to \xi} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} > 0.$$ 

Thus $\mathcal{B}_0^h$ contains no finite Blaschke products. In this section, following [9], it will be shown that there are inner functions in $\mathcal{B}_0^h$. This will then imply the existence of inner functions $\varphi$ such that $C_{\varphi}$ maps $\mathcal{B}_0$ compactly into itself.

\textbf{Lemma 5.10.} Suppose that the function $\eta$ satisfies $\int_0^1 \eta(t)^2 t^{-1} dt = \infty$. Then there exists an increasing function $\sigma$ such that

$$\lim_{t \to 0} \sigma(t) = 0, \quad \sigma(4t) \leq 1.5\sigma(t), \quad \text{and} \quad \int_0^1 \frac{(\eta(t)\sigma(t))^2}{t} dt = \infty.$$ 

\textbf{Proof.} Let $a_0 = 1$, and inductively choose $a_k$ for $k > 1$ such that $0 < 4a_k \leq a_{k-1}$ and $\int_{a_k}^{a_{k-1}} \eta(t)^2 t^{-1} dt \geq 1$. Now define $\sigma(t) = (k + 1)^{-\frac{1}{2}}$ for $a_{k+1} < t \leq a_k$. This has the required properties. \hfill $\square$

\textbf{Theorem 5.11.} Let $\eta$ be a nonnegative increasing function such that, for some $t_0 > 0$,

$$\int_0^1 \frac{\eta(t)^2}{t} dt = \infty \quad \text{and} \quad \eta(4t) \leq 2\eta(t), \quad 0 < t < t_0.$$ 

Let $\sigma$ be the associated function from Lemma 5.10. Then there exists an increasing singular function $f$ defined on $[0, 2\pi]$ and a positive constant $C$ such that, for $h > 0$,

$$|f(x + h) - 2f(x) + f(x - h)| \leq C\eta(h)\sigma(h)h$$

and $f(x + h) - f(x) \geq C^{-1}\sigma(h)h$,

provided $[x - h, x + h] \subset [0, 2\pi]$. 

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Proof. The required function $f$ is constructed as a limit of functions $\{f_n\}$ constructed from $g(x) = \frac{\sin(x) - \sin(2x)}{2}$. The proof is omitted, see [9].

Let $f$ be a function on $[0, 2\pi]$ with continuous periodic extension, and let $G(z) = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) dt$ be the Herglotz integral of $f$.

**Theorem 5.12.** Suppose $\omega(t)$ is positive and nondecreasing for $t > 0$, and that $\omega(4t) \leq 3\omega(t)$ for sufficiently small $t$. If $f$ is continuous and satisfies

$$|f(x + h) - 2f(x) + f(x - h)| \leq \omega(h)h$$

for $h > 0$, then there is a constant $C$ such that

$$(1 - |z|^2)|G''(z)| \leq C\omega(1 - |z|^2).$$

Proof. The proof involves making several estimates and is omitted. See [9] for details.

**Theorem 5.13.** Let $\eta$ be a nonnegative increasing function such that

$$\int_0^1 \frac{\eta(t)^2}{t} dt = \infty$$

and $\eta(4t) \leq 2\eta(t)$, $0 < t < t_0$ for some $t_0 > 0$. Then there exists an inner function $\varphi$ and a constant $C$ such that

$$(1 - |z|^2)|\varphi''(z)| \leq C\eta(1 - |z|^2).$$

A typical function satisfying the hypothesis of Theorem 5.13 is $\eta(t) = |\log(t)|^{-\frac{1}{2}}$.

Proof. Write $\Delta_h f(x) = f(x + h) - f(x)$ and $\Delta_h^3 = f(x + h) - 2f(x) + f(x - h)$. Let $\mu$ be the positive singular measure on $[0, 2\pi]$ with indefinite integral equal to the singular function $f$ from Theorem 5.11, and let

$$F(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

be the Herglotz integral of $\mu$. Now $\varphi(z) = \exp(-F(z))$ satisfies

$$\frac{(1 - |z|^2)|\varphi''(z)|}{1 - |\varphi(z)|^2} \leq \frac{(1 - |z|^2)|F''(z)|}{1 - \exp(-2\text{Re}(F(z)))} \leq \frac{(1 - |z|^2)|F''(z)|}{\text{Re}(F(z))}.$$

Integration by parts shows that $F(z) = iG''(z) - 2\pi K$, where

$$G(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} (f(t) + Kt) dt.$$
Set $K = \frac{f(0) - f(2\pi)}{2\pi}$, so that the periodic extension of $f(t) + K$ is continuous and Theorem 5.12 can be applied. Then $\Delta^2_t f(t) + K$ is continuous, and so the bound $\Delta^2_t f(t) \leq C \sigma(h) \eta(h)\theta$ from Theorem 5.11 along with Theorem 5.12 (applied with $\omega(t) = C \sigma(t) \eta(t)$) gives an upper bound for $|G''(z)|$. Since $|G''(z)| \leq |G'(0)| + |z| \max \{|G''(w)| : |w| \leq |z|\}$, it follows that

$$(1 - |z|^2)|F''(z)| \leq (1 - |z|^2)|zG''(z)| + (1 - |z|^2)|G'(z)| \leq C \eta(1 - |z|^2)\sigma(1 - |z|^2)$$

equation*{1.5}
for all $|z|$ sufficiently close to 1.

Let $z = |z|e^{\theta}$ where $\theta \in [0, 2\pi)$, and assume first that $2\pi \notin [\theta, \theta + (1 - |z|^2)]$. Then $Re(F(z))$ is just the Poisson-Stieltjes integral of $\mu$, so an estimate for it is

$$Re(F(z)) = \int_0^{2\pi} \frac{(1 - |z|^2) d\mu(t)}{|e^{it} - z|^2} \geq \frac{\Delta_t(1 - |z|^2) f(\theta)}{C(1 - |z|^2)} \geq C^{-1} \sigma(1 - |z|^2),$$

where the estimate for $\Delta_t f$ from Theorem 5.11 was used for the last inequality. Thus $\max\{h_1, h_2\} \geq \frac{1 - |z|^2}{2}$, and it follows that the estimate for $Re(F(z))$ above holds too. Combining all the inequalities gives

$$\frac{(1 - |z|^2)|F'(z)|^2}{1 - |\varphi(z)|^2} \leq C \eta(1 - |z|^2).$$

\[\square\]

**Corollary 5.14.** There exists an inner function $I$ such that $C_I$ maps $B_0$ into $B_0$ compactly.

**Proof.** This follows immediately since being in $B_0^h$ characterises compact composition operators by Theorem 4.7, and also since there are inner functions in $B_0^h$ by Theorem 5.13. \[\square\]

Taking $\varphi$ as in Theorem 5.13 with $\overline{\varphi(\mathbb{D})} = \mathbb{D}$, then $C_\varphi$ is a compact operator $B_0 \to B_0$ (Theorem 4.7) such that $C_\varphi(\mathbb{D}) \cap \mathbb{T}$ is infinite.

**Theorem 5.15.** Let $\omega : (0, 1) \to (0, \infty)$ be continuous with $\lim_{t \to 0} \omega(t) = 0$. Then there exists an inner function $I$ such that

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|I'(z)|}{\omega(1 - |I(z)|^2)} = 0.$$ 

**Proof.** Set $I(z) = B(I_0(z))$ where $B$ satisfies the hypothesis of Theorem 5.6 and $I_0 \in B_0^h$ as in Theorem 5.13. Then

$$\frac{(1 - |z|^2)|I'(z)|}{\omega(1 - |I(z)|^2)} = \frac{(1 - |z|^2)|B'(I_0(z))||I_0'(z)|}{\omega(1 - |B(I_0(z))|^2)}.$$
\[ \frac{(1 - |z|^2)|I'(z)|}{1 - |I_0(z)|^2} \to 0, \quad \text{as } |z| \to 1. \]

Schwarz-Pick gives \((1 - |z|^2)|I'(z)| \leq 1 - |I(z)|^2\), but such an \(I\) as in Theorem 5.15 is much sharper. Such an \(I\) decreases hyperbolic distances as much as is liked by using a suitable \(\varphi\).

Corollary 5.8 can be generalised to consider the composition operators induced by inner functions from \(H(w)\) to \(B_0\).

\textbf{Corollary 5.16.} Given \(\epsilon > 0\), \(\exists\) non-constant inner function \(I\) such that composition operator \(C_I\) maps \(H(w)\) into \(B_0\). Moreover \(C_I\) is compact with \(||C_I|| < \epsilon\).

\textit{Proof.} Follows from Corollary 5.8 by composing with the same \(I_0\) as above.

\textbf{References}


