Q-differences and target functions (Joint work with J.K. Langley and J. Meyer)

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The plan for the next 25 minutes is:

- give motivation for the talk;
- introduce the main problem of finding when \( f \circ g - f \) has infinitely many zeros;
- discuss the problem for so-called \( q \)-differences when \( g(z) = qz \) for \(|q| \neq 0, 1\);
- generalize the problem to \( g \) being a non-linear polynomial and then a transcendental entire function;
- generalize again to target functions.
Throughout the talk, we will use standard terminology from value distribution theory:

- $T(r, f)$ denotes the Nevanlinna functional which measures the growth of a meromorphic function $f$ (cf. $\log^+ M(r, f)$ for entire functions).
- The order $\rho$ of a meromorphic function is
  \[
  \rho(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.
  \]
- The lower order $\mu$ replaces $\limsup$ with $\liminf$.
- $\overline{N}(r, 1/f)$ is the integrated counting function of zeros of $f$ with no regard to multiplicity.
As a starting point, consider the following theorem (see Clunie-Eremenko-Rossi 1993, Eremenko-Langley-Rossi 1994, Hinchliffe 2003):

**Theorem**

Let $f$ be transcendental meromorphic in $\mathbb{C}$ with

$$\liminf_{r \to \infty} \frac{T(r, f)}{r} = 0.$$ 

Then $f'$ has infinitely many zeros. Further, if $f$ is entire, then $f'/f$ has infinitely many zeros.
Let

$$\Delta f(z) = f(z + 1) - f(z)$$

and

$$\Delta^{n+1} f(z) = \Delta^n f(z + 1) - \Delta^n f(z)$$

for $n = 1, 2, \ldots$. In view of the previous theorem, it is natural to conjecture whether $\Delta f$ has infinitely many zeros.

**Theorem (Langley, 2009)**

*Let $f$ be transcendental meromorphic in $\mathbb{C}$ of order less than $1/6$. Then at least one of $\Delta f$ and $(\Delta f)/f$ have infinitely many zeros.*
What if $\Delta f$ is replaced with

$$\Delta_q f(z) = f(qz) - f(z)$$

for $q \in \mathbb{C}$ (see Barnett-Halburd-Korhonen-Morgan 2007)? If $q$ is a root of unity, then we can find $f$ such that $\Delta_q f$ has no zeros.

**Theorem (F-Langley-Meyer, 2009)**

Let $q \in \mathbb{C}$ with $|q| > 1$ and let $f$ be transcendental meromorphic with

$$L(f) := \liminf_{r \to \infty} \frac{T(r, f)}{(\log r)^2} = 0.$$ 

Then at least one of $\Delta_q f$ and $(\Delta_q f)/f$ has infinitely many zeros.
The growth rate allowed by this theorem is quite slow, but it is sharp. Fix $q \in \mathbb{C}$ with $|q| > 1$ and define

$$f(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{q^n}\right)^{-1}.$$

Then it is easy to see that

$$f(qz) = \frac{f(z)}{1 - qz}.$$

Hence $(\Delta_q f)/f$ is rational and neither $\Delta_q f$ or $(\Delta_q f)/f$ have infinitely many zeros, while $L(f)$ is positive and finite (and we can make it arbitrarily small by taking $|q|$ arbitrarily large).
Outline of proof

- Since $L(f) = 0$, we can write $f(z) = b_n + c_n z^{\alpha_n} (1 + \delta_n(z))$ on big annuli $A_n$, where $b_n \in \mathbb{C}$, $c_n \in \mathbb{C} \setminus \{0\}$, $\alpha_n \in \mathbb{Z}$ and $\delta_n = o(1)$.

- Show that, by taking subsequences if necessary, $b = \lim_{n \to \infty} b_n \in \mathbb{C} \setminus \{0\}$ and that $f(z) = b$ has finitely many solutions in $\mathbb{C}$.

- Writing $h(z) = f(z) - b$, show that $H = (\Delta_q h)/h$ has infinitely many zeros. If they are poles of $f$, then they are zeros of $(\Delta_q f)/f = H(f - b)/f$. Otherwise, they are zeros of $\Delta_q f$. 

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The methods used in this proof can be used to prove the following generalization.

**Corollary**

Let $f$ be transcendental meromorphic in $\mathbb{C}$, $L(f) = 0$ and $a, b \in \mathbb{C}$ with $|a| \neq 0, 1$. Then at least one of $f(az + b) - f(z)$ and $(f(az + b) - f(z))/f(z)$ has infinitely many zeros.
We have considered zeros of functions of the form $f \circ g - f$ where $g$ is linear. What about if $g$ is a non-linear polynomial? If we first insist that $f$ has finitely many poles, then we have the following result.

**Theorem (F-Langley, 2009)**

Let $f$ be transcendental meromorphic of finite order $\rho(f)$ with finitely many poles and let $g$ be a polynomial of degree $m \geq 2$. Then $F = f \circ g - f$ has infinitely many zeros, and if $\rho > 0$, then the exponent of convergence of the zeros of $F$ is $\rho(F) = m\rho(f)$. 
A reminder of the exponent of convergence

Given a sequence \((a_n)\) in \(\mathbb{C}\), arranged in order of non-decreasing modulus, let \(n(r)\) denote the number of \(a_n\) in \(|z| \leq r\) and set

\[
N(r) = \int_0^r \frac{n(t)}{t} \, dt.
\]

The exponent of convergence \(\lambda\) of the sequence \((a_n)\) is

\[
\lambda = \limsup_{r \to \infty} \frac{\log N(r)}{\log r} = \limsup_{r \to \infty} \frac{\log n(r)}{\log r},
\]

and is also the infimum of \(c > 0\) such that \(\sum |a_n|^{-c}\) converges.

Simple example: \(a_n = n\) for \(n \in \mathbb{N}\) has exponent of convergence 1.
In the case where $f$ is allowed to have infinitely many poles, there is the following result.

**Theorem (F-Langley 2009)**

Let $f$ be transcendental meromorphic of finite order $\rho(f)$, $g$ be a polynomial of degree $m \geq 2$ and $F = f \circ g - f$. If $0 < \rho(f) < 1/m$, or if $\rho = 0$ and $m \geq 4$, then $F$ has infinitely many zeros. If $\rho = 0$, then the equation $f(g(z)) = f(z)$ has infinitely many solutions in $\mathbb{C}$.
When $g$ is transcendental

First, note that if $g$ is transcendental entire with no fixed points and $f = R \circ g^n$ for some $n \in \mathbb{N}$, where $R$ is a Möbius transformation, then

$$F = f \circ g - f = R \circ g^{n+1} - R \circ g^n$$

has no zeros since if $w$ a zero of $F$, then $g^n(w)$ is a fixed point of $g$. This gives a hint to the following theorem, which says that if $F$ has too few zeros, then $f$ and $g$ must be of a certain special form.
When \( g \) is transcendental

**Theorem (F-Langley 2009)**

Let \( f \) be transcendental meromorphic of finite order and \( g \) transcendental entire of finite lower order. Assume that there exists a set \( E \subseteq [1, \infty) \) of positive lower logarithmic density such that \( F = f \circ g - f \) and \( f \) satisfy

\[
\overline{N}(r, 1/F) + T(r, f) = O(T(r, g))
\]

on \( E \). Then there exist a Möbius transformation \( R \) and polynomials \( P, S \) such that \( f = R \circ g \) and

\[
g(z) = z + S(z)e^{P(z)}.
\]

If \( f \) has finitely many poles, then \( f = ag + b \) for \( a, b \in \mathbb{C} \).
When $g$ is transcendental

- The hypotheses imply that we must have the growth of $f$ being controlled by the growth of $g$, at least on a set of positive lower logarithmic density $E$; that is, $E$ must satisfy
\[ \liminf_{r \to \infty} \frac{\int_{E \cap [1, r]} dt}{t \log r} > 0. \]

- We also must have the exponent of convergence of the zeros of $F$ being at most the order of $g$, when restricted to the set $E$.

- It then follows that $g$ must have finitely many fixed points, and $f$ is just $g$ post-composed by a Möbius transformation.

- Conversely, if $f$ and $g$ are not of this form, then the exponent of convergence of the zeros of $F$ must be larger than the order of $g$.  

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The previous theorem is a special case of the following theorem, which improves upon results of Katajamäki, Kinnunen and Laine. This theorem considers zeros of $F = f \circ g - Q$ where $Q$ is called a target function.
Theorem (F-Langley, 2009)

The hypotheses: let $f$, $g$ and $Q$ be meromorphic in $\mathbb{C}$ with the following properties:

- $f$ is transcendental meromorphic of finite order;
- $g$ is transcendental entire of finite lower order;
- there exists a set $E \subseteq [1, \infty)$ of positive lower logarithmic density such that $Q$ and $F = f \circ g - Q$ satisfy

\[
T(r, Q) + \overline{N}(r, 1/F) = O(T(r, g))
\]

on $E$. 
Theorem (F-Langley, 2009, continued)

Then at least one of the following two conclusions occurs:

- there exists a rational function \( R \) such that \( f - R \) has finitely many zeros, \( Q = R \circ g \), and this conclusion always occurs if \( f \) has finitely many poles;

- there exist rational functions \( A, B, C \) such that \( f \) solves the Riccati equation \( y' = A + By + Cy^2 \) and \( Q' = g'(A \circ g + (B \circ g)Q + (C \circ g)Q^2) \) so that locally we may write \( Q = w \circ g \) for some solution \( w \) of the Riccati equation above.

Further, if \( T(r, Q) = o(T(r, g)) \) on \( E \), then \( Q \) must be constant.
It is worth remarking that if \( f \) has infinitely many poles, then the second case can occur with the local solution \( w \) not meromorphic in the plane (e.g. it could involve a \( z^{1/2} \)).

If \( Q = w \circ g \) with \( w \) meromorphic in the plane, then since the growth of \( Q \) is controlled by the growth of \( g \), a well-known result of Clunie implies that \( w \) must be a rational function.

A detailed discussion of the proof of the theorem would need a whole new talk...
Questions

- It seems very plausible that both $\Delta_q f$ and $(\Delta_q(f))/f$ have infinitely many zeros if $L(f) = 0$. Is this true?
- If $L(f) = 0$, $|q| = 1$ and $q$ is not a root of unity, must $f(qz) - f(z)$ have infinitely many zeros?
- If $\rho(f) = 0$ and $g$ is a polynomial of degree 2 or 3, show that $f \circ g - f$ has infinitely many zeros.
- Can one say anything about quasiregular target functions? For example, Bergweiler has shown that if $f$ and $g$ are quasiregular in $\mathbb{R}^n$ with essential singularities at infinity and $Q(z) = z$, then $f \circ g - Q$ has infinitely many zeros.