Quasiregular dynamics on the $n$-sphere

A. Fletcher

$^1$University of Glasgow/Warwick

Analysis Seminar, September 2009
In this talk, I will

- cover the basic notions of complex dynamics,
- introduce quasiregular mappings as higher dimensional analogues of analytic mappings,
- talk a bit about quasiregular dynamics, including the main theorem of the talk involving the boundary of the escaping set for quasiregular mappings of polynomial type,
- discuss further questions and directions of research.
Complex dynamics deals with the iteration of functions in $\mathbb{C}$. These functions can be polynomials, entire functions or meromorphic functions. We write $f^n$ for the $n$-fold iterate of $f$, e.g. $f^2 = f \circ f$.

The Fatou set $F(f)$ is where the iterates behave "nicely", the Julia set $J(f)$ is where the iterates behave "chaotically" and the escaping set $I(f)$ is

$$I(f) := \{ z \in \mathbb{C} : \lim_{n \to \infty} f^n(z) = \infty \}.$$ 

A simple example is $f(z) = z^2$. Then $J(f)$ is the unit circle, $F(f)$ is the interior and exterior components of the complement of the unit circle and $I(f)$ is the exterior component of the unit circle.
A point $x \in \mathbb{C}$ is in $F(f)$ if there is a neighbourhood $U$ of $x$ such that the family $\{f^n|_U\}$ is normal. Recall a normal family is one for which every sequence contains a convergent subsequence.

Similarly, a point $x \in \mathbb{C}$ is in $J(f)$ if for every neighbourhood $V$ of $x$, the family $\{f^n|_V\}$ is not normal.

It is well-known (Eremenko, Dominguez) that $J(f) = \partial I(f)$.

The Julia set has certain properties, eg. it is a perfect set and it is the closure of the repelling periodic points.
**Quasiregular mappings**

- Quasiregular mappings are a natural generalization of analytic functions to higher dimensions. They are defined by conditions on regularity and distortion.

Let $G \subseteq \mathbb{R}^n$ be a domain with $n \geq 2$. A mapping $f : G \to \mathbb{R}^n$ is called quasiregular if it is in the Sobolev space $W^{1,n}_{loc}(G)$ and there exists $K \in [1, \infty)$ such that

$$|f'(x)|^n \leq K|J_f(x)|$$

almost everywhere, where $J_f$ is the Jacobian of the mapping $f$.

The smallest $K$ in (1) is the outer dilatation $K_O(f)$ of $f$. If $f$ is quasiregular, then we also have

$$J_f(x) \leq K' l(f'(x))^n$$

almost everywhere, for some $K' \geq 1$, where $l(f'(x)) = \inf_{|h|=1} |f'(x)h|$. The smallest $K' \geq 1$ in (2) is called the inner dilatation $K_I(f)$ of $f$. 

---

A. Fletcher

Quasiregular dynamics on the $n$-sphere
Quasiregular mappings are a natural generalization of analytic functions to higher dimensions. They are defined by conditions on regularity and distortion.

Let $G \subseteq \mathbb{R}^n$ be a domain with $n \geq 2$. A mapping $f : G \rightarrow \mathbb{R}^n$ is called quasiregular if it is in the Sobolev space $W^{1,n}_{loc}(G)$ and there exists $K \in [1, \infty)$ such that

$$|f'(x)|^n \leq K J_f(x)$$

almost everywhere, where $J_f$ is the Jacobian of the mapping $f$.

The smallest $K$ in (1) is the outer dilatation $K_O(f)$ of $f$. If $f$ is quasiregular, then we also have

$$J_f(x) \leq K' l(f'(x))^n$$

almost everywhere, for some $K' \geq 1$, where $l(f'(x)) = \inf_{|h|=1} |f'(x)h|$. The smallest $K' \geq 1$ in (2) is called the inner dilatation $K_I(f)$ of $f$. 

A. Fletcher  Quasiregular dynamics on the $n$-sphere
Properties of quasiregular mappings

For further information and references on quasiregular mappings, see Rickman’s book.

- The maximal dilatation $K(f)$ is the larger of $K_O(f)$ and $K_I(f)$. A mapping is called $K$-quasiregular if $K(f) \leq K$.
- A bit of linear algebra shows that $K_O(f) \leq K_I(f)^{n-1}$ and $K_I(f) \leq K_O(f)^{n-1}$. Therefore $K_O(f) = K_I(f)$ when $n = 2$.
- A quasiregular homeomorphism is called quasiconformal. When $n = 2$, a quasiregular mapping $f$ is always of the form $f = g \circ h$ where $h$ is quasiconformal and $g$ is analytic.
Some more properties of quasiregular mappings

- When \( n \geq 3 \), every 1-quasiregular mapping is a restriction of a Möbius transformation or a constant (Reshetnyak, 1967).
- Non-constant quasiregular mappings are discrete and open (Reshetnyak 1967/8).
- There is a quasiregular analogue of Picard’s theorem. Given \( n \geq 2 \) and \( K \geq 1 \), there exists an integer \( C(n, K) \) such that if \( f \) is \( K \)-quasiregular on \( \mathbb{R}^n \) and omits \( C(n, K) \) distinct values in \( \mathbb{R}^n \), then \( f \) is constant (Rickman, 1980). By the decomposition of quasiregular mappings when \( n = 2 \), \( C(2, K) = 2 \).
The iterates of quasiregular mappings are again quasiregular and properties such as the existence of periodic points have been investigated (Bergweiler, 2006 and Siebert, 2006).

A uniformly quasiregular mapping has the property that all its iterates have a common bound on the maximal dilatation. Most work on the dynamics of quasiregular mappings has been done in the uniformly quasiregular case (Iwaniec-Martin, Hinkkanen-Martin-Mayer, Peltonen, Kangaslampi).

Some concepts of complex dynamics are difficult to extend to quasiregular mappings without uniform quasiregularity, but the escaping set still makes sense. Bergweiler-F-Langley-Meyer (2009) proved that with a certain growth condition (e.g. an essential singularity at infinity), the escaping set of a quasiregular mapping is non-empty.
Dynamics of quasiregular mappings

- The iterates of quasiregular mappings are again quasiregular and properties such as the existence of periodic points have been investigated (Bergweiler, 2006 and Siebert, 2006).

- A uniformly quasiregular mapping has the property that all its iterates have a common bound on the maximal dilatation. Most work on the dynamics of quasiregular mappings has been done in the uniformly quasiregular case (Iwaniec-Martin, Hinkkanen-Martin-Mayer, Peltonen, Kangaslampi).

- Some concepts of complex dynamics are difficult to extend to quasiregular mappings without uniform quasiregularity, but the escaping set still makes sense. Bergweiler-F-Langley-Meyer (2009) proved that with a certain growth condition (e.g. an essential singularity at infinity), the escaping set of a quasiregular mapping is non-empty.
The main theorem of the talk

General overall aim: show that the boundary of the escaping set has properties analogous to the Julia set, when the Julia set is not defined.

**Theorem (F-Nicks, 2009)**

Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be $K$-quasiregular of polynomial type. If the degree of $f$ is greater than $K_I$, then $I(f)$ is a non-empty open set and $\partial I(f)$ is perfect (i.e. no isolated points).
General overall aim: show that the boundary of the escaping set has properties analogous to the Julia set, when the Julia set is not defined.

**Theorem (F-Nicks, 2009)**

Let \( n \geq 2 \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) be \( K \)-quasiregular of polynomial type. If the degree of \( f \) is greater than \( K_I \), then \( I(f) \) is a non-empty open set and \( \partial I(f) \) is perfect (i.e. no isolated points).
A quasiregular mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be of polynomial type if $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, whereas $f$ has an essential singularity at infinity if this limit does not exist.

We consider the point at infinity to be contained in $I(f)$ if $f$ is of polynomial type, but not if $f$ has an essential singularity at infinity.

The degree of a polynomial type mapping may be thought of as a generalization of the degree of a polynomial. It can be defined by

$$\text{deg}(f) := \sup_{y \in \mathbb{R}^n} |f^{-1}(y)|,$$

that is, the maximal number of pre-images of any value in $\mathbb{R}^n$. It is well known that $f$ is of polynomial type if and only if (3) is finite.
Remarks on the theorem

The theorem is sharp as the following example shows. Let $n \geq 2$, $k \in \mathbb{N}$ and consider the mapping

$$f : (r, \varphi, y) \mapsto (r, k\varphi, y)$$

in cylindrical coordinates in $\mathbb{R}^n$ (i.e. $y \in \mathbb{R}^{n-2}$). The branch set of $f$ is the $(n - 2)$-dimensional hyperplane defined by $r = 0$. The degree of $f$ can be shown to be $k$, and further $K_I(f) = k$. However $|f(x)| = |x|$ for all $x \in \mathbb{R}^n$, and so $I(f) \cap \mathbb{R}^n$ is empty.

Let $f$ satisfy the hypotheses of the theorem. Then

1. for any $k \geq 2$ we have $I(f^k) = I(f)$.
2. The family of iterates $\{f^k : k \in \mathbb{N}\}$ is equicontinuous on $I(f)$ and not equicontinuous at any point of $\partial I(f)$, with respect to the spherical metric on $\mathbb{R}^n$.
3. $\partial I(f)$ is infinite.
4. $I(f), \partial I(f)$ and $\mathbb{R}^n \setminus I(f)$ are completely invariant.
5. $I(f)$ is connected.
The theorem is sharp as the following example shows. Let \( n \geq 2, k \in \mathbb{N} \) and consider the mapping 
\[ f : (r, \varphi, y) \mapsto (r, k\varphi, y) \]
in cylindrical coordinates in \( \mathbb{R}^n \) (i.e. \( y \in \mathbb{R}^{n-2} \)). The branch set of \( f \) is the \((n - 2)\)-dimensional hyperplane defined by \( r = 0 \). The degree of \( f \) can be shown to be \( k \), and further \( K_I(f) = k \). However \( |f(x)| = |x| \) for all \( x \in \mathbb{R}^n \), and so \( I(f) \cap \mathbb{R}^n \) is empty.

Let \( f \) satisfy the hypotheses of the theorem. Then

1. for any \( k \geq 2 \) we have \( I(f^k) = I(f) \).
2. The family of iterates \( \{f^k : k \in \mathbb{N}\} \) is equicontinuous on \( I(f) \) and not equicontinuous at any point of \( \partial I(f) \), with respect to the spherical metric on \( \mathbb{R}^n \).
3. \( \partial I(f) \) is infinite.
4. \( I(f), \partial I(f) \) and \( \mathbb{R}^n \setminus \overline{I(f)} \) are completely invariant.
5. \( I(f) \) is connected.
Idea of the proof

The first ingredient is the local Hölder property of quasiregular mappings, due to Martio. Let $f : G \rightarrow \mathbb{R}^n$ be quasiregular and non-constant, and let $x \in G$. Then there exist positive constants $r$ and $C$ such that for $|x - y| < r$,

$$|f(y) - f(x)| \leq C|y - x|^{\alpha}$$

where

$$\alpha = \left( \frac{i(x, f)}{K_i(f)} \right)^{1/(n-1)}.$$ 

Here $i(x, f)$ is the local topological index of $f$ at $x$, and if a point $x$ is such that $f^{-1}(x) = \{x\}$, then $i(x, f) = \deg(f)$.
Idea of the proof

- We therefore conjugate $f$ by a Möbius transformation swapping 0 and infinity and apply the Martio result at 0.
- This leads to the existence of a constant $R > 0$ such that

$$|f(x)| > 2|x|$$

for $|x| > R$. Therefore $f^k(x) \to \infty$ as $k \to \infty$.
- This implies that infinity is an attracting fixed point of $f$ and the basin of attraction includes $\{x \in \mathbb{R}^n : |x| > R\}$.
- Consequently we have that $I(f)$ is non-empty, and further, that $I(f)$ contains a neighbourhood of infinity.
- This fact, together with the fact that $f$ is continuous, quickly leads to the fact that $I(f)$ is open and hence cannot contain any isolated points. One just then has to show that $I(f)^c$ contains no isolated points to finish the proof.

A. Fletcher  Quasiregular dynamics on the $n$-sphere
Further question on polynomial-type mappings

- An important property of the Julia set of analytic functions is the expanding property. That is, let $x \in J(f)$ and $U$ be a neighbourhood of $x$. Then

$$\mathbb{C} \setminus \{y\} \subset \bigcup_{j=1}^{\infty} f^j(U),$$

where $y \in \mathbb{C}$ (again, consider $f(z) = z^2$).

- One would conjecture the existence of a constant $C(n, K)$ such that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $K$-quasiregular mapping of polynomial type satisfying $K_l < \deg(f)$ and $x \in \partial I(f)$, $U$ is a neighbourhood of $x$, then there exist $C(n, K)$ points $y_1, \ldots, y_C$ such that

$$\mathbb{R}^n \setminus \{y_1, \ldots, y_C\} \subset \bigcup_{j=1}^{\infty} f^j(U).$$
These questions on the boundary of the escaping set can also be asked for quasiregular mappings with essential singularities at infinity, although differences analogous to those between polynomial and entire functions are to be expected.

Zorich type mappings are generalizations of the exponential functions to higher dimensions. Easiest way to explain them is via pictures...

As with $e^z$, it appears that the escaping set of Zorich maps are made up of hairs. It seems natural to conjecture that $\partial I(f)$, for at least some of these Zorich maps, is the whole of $\mathbb{R}^n$, analogous to the fact that $J(e^z) = \mathbb{C}$, and that $\partial I(f)$ is then perfect.
Other questions

- Is $\partial I(f)$ the closure of the repelling periodic points for quasiregular mappings?
- Is $\partial I(f)$ perfect for general quasiregular $f$?
- We can show that $\partial I(f)$ is perfect for uniformly quasiregular mappings, but are there any uqr mappings with essential singularities at infinity?
- A related question: in dimension 2 one has the factorisation $f = g \circ h$ for a quasiregular mapping $f$ into a quasiconformal map $h$ and an analytic map $g$. Can one produce a similar factorisation in higher dimensions involving uniformly quasiregular mappings instead of analytic mappings? Yes (Martin-Peltonen), if $f$ is defined at infinity.