THE ESCAPING SET OF A QUASIREGULAR MAPPING

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Abstract. We show that if the maximum modulus of a quasiregular mapping \( f : \mathbb{R}^N \to \mathbb{R}^N \) grows sufficiently rapidly then there exists a non-empty escaping set \( I(f) \) consisting of points whose forward orbits under iteration of \( f \) tend to infinity. We also construct a quasiregular mapping for which the closure of \( I(f) \) has a bounded component. This stands in contrast to the situation for entire functions in the complex plane, for which all components of the closure of \( I(f) \) are unbounded, and where it is in fact conjectured that all components of \( I(f) \) are unbounded.


1. Introduction

In the study [1] of the dynamics of nonlinear entire functions \( f : \mathbb{C} \to \mathbb{C} \) considerable recent attention has focussed on the escaping set

\[ I(f) = \{ z \in \mathbb{C} : \lim_{n \to \infty} f^n(z) = \infty \}, \]

where \( f^1 = f, f^{n+1} = f \circ f^n \) denote the iterates of \( f \). Eremenko [5] proved that if \( f \) is transcendental then \( I(f) \neq \emptyset \) and indeed that, in keeping with the nonlinear polynomial case [18], the boundary of \( I(f) \) is the Julia set \( J(f) \). The proof in [5] that \( I(f) \) is non-empty is based on the Wiman-Valiron theory [6].

For transcendental entire functions \( f \), Eremenko went on to prove in [5] that all components of the closure of \( I(f) \) are unbounded, and to conjecture that the same is true of \( I(f) \) itself. For entire functions with bounded postcritical set this conjecture was proved by Rempe [12], and for the general case it was shown by Rippon and Stallard [16] that \( I(f) \) has at least one unbounded component.

In the meromorphic case the set \( I(f) \) was first studied by Dominguez [4], who proved that again \( I(f) \neq \emptyset \) and \( \partial I(f) = J(f) \). For meromorphic \( f \) it is possible that all components of \( I(f) \) are bounded [4], and the closure of \( I(f) \) may have bounded components even if \( f \) has only one pole [4, p.229]. On the other hand \( I(f) \) always has at least one unbounded component if the inverse function \( f^{-1} \) has a direct transcendental singularity over infinity: this was proved by Bergweiler, Rippon and Stallard [3] by developing an analogue of the Wiman-Valiron theory in the presence of a direct singularity.

The present paper is concerned with the escaping set for quasiregular mappings \( f : \mathbb{R}^N \to \mathbb{R}^N \) [15], which represent a natural counterpart in higher real dimensions of analytic functions in the plane, and exhibit many analogous properties, a highlight among these being Rickman’s Picard theorem for entire quasiregular maps.
For the precise definition and further properties of quasiregular mappings we refer the reader to Rickman’s text [15]. Now the iterates of an entire quasiregular map are again quasiregular, and properties such as the existence of periodic points were investigated in [2, 17]. Further, there is increasing interest in the dynamics of quasiregular mappings on the compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, although attention has been restricted to mappings which are uniformly quasiregular in the sense that all iterates have a common bound on their dilatation: see [8, Section 21] and [7]. In the absence of this uniform quasiregularity there are evidently some difficulties in extending some concepts of complex dynamics to quasiregular mappings in general, but the escaping set $I(f)$ makes sense nevertheless, and we shall prove the following theorem.

**Theorem 1.1.** Let $N \geq 2$ and $K > 1$. Then there exists $J > 1$, depending only on $N$ and $K$, with the following property.

Let $R > 0$ and let $f : D_R \to \mathbb{R}^N$ be a $K$-quasiregular mapping, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set

$$B_R = \{ x \in \mathbb{R}^N : R \leq |x| < \infty \}.$$

Assume that $f$ satisfies

$$\liminf_{r \to \infty} \frac{M(r, f)}{r} \geq J,$$

where $M(r, f) = \max\{|f(x)| : |x| = r\}$, and define the escaping set by

$$I(f) = \{ x \in \mathbb{R}^N : \lim_{n \to \infty} f^n(x) = \infty \}, \quad f^1 = f, \quad f^{n+1} = f \circ f^n.$$

Then $I(f)$ is non-empty. If, in addition, $f$ is $K$-quasiregular on $\mathbb{R}^N$ then $I(f)$ has an unbounded component.

The proof of Theorem 1.1 is based on the approach of Dominguez [4], as well as that of Rippon and Stallard [16]. A key role is played also by the analogue of Zalcman’s lemma [19, 20] developed for quasiregular mappings by Miniowitz [10] (see §2). It seems worth observing that in Theorem 1.1 the hypothesis (2) cannot be replaced by

$$\liminf_{r \to \infty} \frac{M(r, f)}{r} > 1,$$

as the following example shows. Take cylindrical polar coordinates $r \cos \theta, r \sin \theta, x_3$ in $\mathbb{R}^3$, let $\lambda > 0$ and and let $f$ be the mapping defined by

$$0 \to 0, \quad (re^{i\theta}, x_3) \to (re^{\lambda \cos \theta + i(\theta + \pi)}, x_3).$$

Then $f^2$ is given by

$$(re^{i\theta}, x_3) \to (re^{\lambda \cos \theta + i(\theta + 2\pi)}, x_3)$$

and so is the identity, while since $f$ is $C^1$ on $\mathbb{R}^3 \setminus \{0\}$ and satisfies $f(2x) = 2f(x)$ it is easy to see that $f$ is quasiconformal on $\mathbb{R}^3$. On the other hand if $f : \mathbb{R}^N \to \mathbb{R}^N$ is quasiregular with an essential singularity at infinity, then $M(r, f)/r \to \infty$ as $r \to \infty$ (see, for example, [2, Lemma 3.4]) so that (2) holds with any $J > 1$.

Next, we show in §6 that there exists a quasiregular mapping $f$ on $\mathbb{R}^2$ with an essential singularity at infinity, such that the closure of $I(f)$ has a bounded component. Thus while the result of [16] that $I(f)$ has at least one unbounded component extends to quasiregular mappings by Theorem 1.1, Eremenko’s theorem [5] that all components of the closure of $I(f)$ are unbounded does not.
We remark finally that it is easy to show that if $f$ is quasimeromorphic with infinitely many poles in $\mathbb{R}^N$ then $I(f)$ is non-empty, and for completeness we outline how this is proved in §7, using the “jumping from pole to pole” method [3, 4].

2. Theorems of Rickman and Miniowitz

Let $G$ be a domain in $\mathbb{R}^N$. A continuous mapping $f : G \to \mathbb{R}^N$ is called quasiregular [15] if $f$ belongs to the Sobolev space $W^{1}_{N,\text{loc}}(G)$ and there exists $K \in [1, \infty)$ such that $|f'(x)|^N \leq KJ_f$ a.e. in $G$. Moreover, $f$ is called $K$-quasiregular if its inner and outer dilations do not exceed $K$: for the details and equivalent definitions we refer the reader to [15]. Furthermore, if $f : G \to \mathbb{R}^N = \mathbb{R}^N \cup \{\infty\}$ is continuous then $f$ is called quasimeromorphic [9] if each $x \in G$ has a neighbourhood $U_x$ such that either $f$ or $g \circ f$ is a quasiregular map of $U_x$ into $\mathbb{R}^N$, where $g$ is a sense-preserving Möbius map of $\mathbb{R}^N$ with $g(\infty) \in \mathbb{R}^N$.

Rickman proved [13, 15] that given $N \geq 2$ and $K \geq 1$ there exists an integer $C(N, K)$ such that if $f$ is $K$-quasiregular on $\mathbb{R}^N$ and omits $C(N, K)$ distinct values $a_j \in \mathbb{R}^N$ then $f$ is constant. Here $C(2, K) = 2$ because a quasiregular mapping in $\mathbb{R}^2$ may be written as the composition of a quasiconformal mapping with an entire function, but for $N \geq 3$ this integer $C(N, K)$ in general exceeds 2 [14, 15].

Miniowitz [10] established for quasiregular mappings the following direct analogue of Zalcman’s lemma [19, 20]. A family $F$ of $K$-quasiregular mappings on the unit ball $B^N$ of $\mathbb{R}^N$ is not normal if and only if there exist

$$f_n \in F, \quad x_n \in B^N, \quad x_n \to \hat{x} \in B^N, \quad \rho_n \to 0+$$

and a non-constant $K$-quasiregular mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ with the property that $f_n(x_n + \rho_n x) \to f(x)$ locally uniformly in $\mathbb{R}^N$, with respect to the spherical distance $\chi(x, y)$ on $\mathbb{R}^N$. Using this she established the following analogue of Montel’s theorem, in which $C(N, K)$ is the integer from Rickman’s theorem [13].

**Theorem 2.1** ([10]). Let $N \geq 2, K > 1, \varepsilon > 0$ and let $D$ be a domain in $\mathbb{R}^N$. Let $F$ be a family of functions $K$-quasiregular on $D$ with the following property. Each $f \in F$ omits $q = C(N, K)$ values $a_1(f), \ldots, a_q(f)$ on $D$, which may depend on $f$ but satisfy

$$\chi(a_j(f), a_k(f)) \geq \varepsilon \quad \text{for} \quad j \neq k.$$ 

Then $F$ is normal on $D$.

Theorem 2.1 leads at once to the following standard lemma of Schottky type.

**Lemma 2.1.** Let $N \geq 2$ and $K > 1$. Then there exists $Q > 2$ with the following property. Let $f$ be $K$-quasiregular on the set $\{x \in \mathbb{R}^N : 1 < |x| < 4\}$ such that $f$ omits $q = C(N, K)$ values $y_1, \ldots, y_q$, with

$$|y_j| = 4^{j-1}, \quad j = 1, \ldots, q.$$ 

If $\min\{|f(x)| : |x| = 2\} \leq 2$ then $\max\{|f(x)| : |x| = 2\} \leq Q$.

3. Two Lemmas Needed for Theorem 1.1

We need the following two facts, the first of which is from Newman’s book [11, Exercise, p.84]:

**Lemma 3.1.** Let $G$ be a continuum in $\overline{\mathbb{R}^N} = \mathbb{R}^N \cup \{\infty\}$ such that $\infty \in G$, and let $H$ be a component of $\mathbb{R}^N \cap G$. Then $H$ is unbounded.
This leads on to the second fact we need:

**Lemma 3.2.** Let $E$ be a continuum in $\mathbb{R}^N$ such that $\infty \in E$, and let $g : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous open mapping. Then the preimage

$$g^{-1}(E) = \{ x \in \mathbb{R}^N : g(x) \in E \}$$

cannot have a bounded component.

For completeness we give a proof of Lemma 3.2 in §8.

4. **An analogue of Bohr’s theorem**

Let $f : D_R \to \mathbb{R}^N$ be $K$-quasiregular, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set $B_R$ in (1), and assume that $f$ satisfies (2) for some $J > 1$. For $0 \leq r < s \leq \infty$

$$A(r, s) = \{ x \in \mathbb{R}^N : r < |x| < s \}.$$

Using (2) choose $s_0 > R$ such that

$$M(r, f) > M(R, f) \quad \text{for all} \quad r \geq s_0.$$

Then $M(r, f)$ is strictly increasing on $[s_0, \infty)$ because if $s_0 \leq r_1 < r_2 < \infty$ and $M(r_2, f) \leq M(r_1, f)$ then $|f(x)|$ has a local maximum at some $\hat{x} \in A(R, r_2)$, which contradicts the fact that non-constant quasiregular mappings send open sets to open sets [15, Theorem 4.1, p.16]. Following Dominguez [4] we establish a lemma analogous to Bohr’s theorem.

**Lemma 4.1.** Let $c = 1/2Q$, where $Q$ is the constant of Lemma 2.1. Then for all sufficiently large $\rho$ there exists $L \geq cM(\rho/2, f)$ such that

$$S(0, L) = \{ x \in \mathbb{R}^N : |x| = L \} \subseteq f(A(R, \rho)).$$

**Proof.** Using (2) let $\rho$ be so large that

$$\rho > 4R \quad \text{and} \quad S = cM(\rho/2, f) > 2T = 4M(R, f),$$

and assume that the assertion of the lemma is false for $\rho$. Then for $j = 1, \ldots, q$, where $q = C(N, K)$ is the integer from Rickman’s Picard theorem [13] (see §2), there exists $a_j \in \mathbb{R}^N$ with

$$|a_j| = 4^{j-1}S \quad \text{and} \quad a_j \notin f(A(R, \rho)).$$

Furthermore, there exists $x_1 \in A(R, \rho/2)$ such that $|f(x_1)| = S$. To see this join a point $x_0$ on $S(0, \rho/2)$ such that $|f(x_0)| = M(\rho/2, f)$ to $S(0, R)$ by a radial segment and use (4) and the fact that $c < 1$. Let $G$ be the component of the set

$$\{ x \in \mathbb{R}^N : T < |f(x)| < 2S \}$$

which contains $x_1$. Then $G \subseteq A(R, \infty)$ by (4). Suppose first that $G \subseteq A(R, \rho/2)$. Then the closure $\overline{G}$ of $G$ lies in $A(R, \rho)$, by (4) again. Choose a geodesic $\sigma \subseteq S(0, S)$ joining $f(x_1)$ to $a_1$. Let

$$\mu = \inf\{|f(x) - a_1| : x \in \overline{G}, f(x) \in \sigma\}$$

and take $\zeta_0 \in \overline{G}$ with $f(\zeta_0) \in \sigma$ and $|f(\zeta_0) - a_1| \to \mu$. Then we may assume that $\zeta_0 \to \hat{\zeta} \in \overline{G}$, and we have $f(\hat{\zeta}) \in \sigma$ and so $\zeta \in G$. But then the open mapping theorem forces $\mu = |f(\hat{\zeta}) - a_1| = 0$, which contradicts (5).

Thus $G \not\subseteq A(R, \rho/2)$ and this implies using (4) again that there must exist $x_2$ on $S(0, \rho/2)$ such that $|f(x_2)| \leq 2S$. By (4) and (5) the function $g(x) = f(x\rho/4)/S$
is $K$-quasiregular on $A(1,4)$, and omits the $q$ values $y_j = y_j/S$, which satisfy $|y_j| = 4^{j-1}$. Since $|g(4x/\rho)| \leq 2$, Lemma 2.1 implies that $|g(x)| \leq Q$ for $|x| = 2$, which gives

$$M(\rho/2, f) \leq QS = QcM(\rho/2, f) = \frac{M(\rho/2, f)}{2},$$

a contradiction. \hfill $\square$

5. Proof of Theorem 1.1

Again let $f : D_R \to \mathbb{R}^N$ be $K$-quasiregular, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set $B_R$ in (1), but this time assume that $f$ satisfies (2) for some large positive $J$. Retain the notation of §4. Following Dominguez’ method [4] let $\rho_0 > R$ be so large that every $\rho \geq \rho_0$ satisfies the conclusion of Lemma 4.1 and further that, with the same constant $c$ as in Lemma 4.1,

$$cM(\rho/2, f) > 4\rho > \rho > M(R, f) \quad \forall \rho \geq \rho_0,$$

which is possible by (2) and the assumption that $J$ is large. Fix $\rho \geq \rho_0$.

Lemma 5.1. There exist bounded open sets $G_0, G_1, \ldots$ with the following properties.

(i) The set $\mathbb{R}^N \setminus G_n$ has two components, namely

$$\tilde{G}_n = \overline{B(0,R)} = \{ x \in \mathbb{R}^N : |x| \leq R \}$$

and $G_n^* = A_n$, which satisfies $\infty \in A_n$.

(ii) We have

$$\{ x \in \mathbb{R}^N : R < |x| \leq 2^n \rho \} \subseteq G_n.$$  

(iii) The sets $G_n, A_n$ and $\gamma_n = \partial A_n$ satisfy

$$\gamma_{n+1} \subseteq f(\gamma_n) \quad \text{and} \quad f(G_n) \cap A_{n+1} = \emptyset.$$  

Proof. The open sets $G_n$ will be constructed inductively. We begin by setting $G_0 = A(R, \rho')$ for some $\rho' > \rho$, so that (7) obviously is satisfied for $n = 0$. It remains to show how to construct $G_{n+1}$ given the existence of $G_0, \ldots, G_n$ for some $n \geq 0$. The fact that $f$ maps open sets to open sets gives

$$\partial f(G_n) \subseteq f(\partial G_n) = f(S(0,R)) \cup f(\gamma_n),$$

using (i) and the definition $\gamma_n = \partial A_n$. By Lemma 4.1, (6) and (7) there exists

$$T_n \geq cM(2^n, f) > 2^{n+2} \rho' \quad \text{with} \quad S(0,T_n) \subseteq f(A(R,2^n \rho')) \subseteq f(G_n).$$

Now $f(G_n)$ is a bounded open set, so let $A_{n+1}$ be the component of $\mathbb{R}^N \setminus f(G_n)$ which contains $\infty$ and set

$$\gamma_{n+1} = \partial A_{n+1}.$$  

Then by (10) we have

$$\gamma_{n+1} \subseteq A_{n+1} \subseteq A(2^{n+2} \rho, \infty),$$

and (6), (9) and (11) imply the first assertion of (8). Let

$$G_{n+1} = \mathbb{R}^N \setminus (\overline{B(0,R)} \cup A_{n+1}).$$

Then (i) is satisfied with $n$ replaced by $n + 1$, and the second assertion of (8) follows from the definition of $A_{n+1}$. Finally (12) shows that (7) is satisfied with $n$ replaced by $n + 1$, and so the induction is complete. \hfill $\square$
Lemma 5.2. Let $w \in \gamma_n$. Then there exists $z_n \in \gamma_0$ with $f^n(z_n) = w$ and

\[ f^n(z_n) \in \gamma_m \quad \text{for} \quad m = 0, \ldots, n. \tag{13} \]

Proof. This is easily proved using induction and (8).

Now take a sequence of points $z_n \in \gamma_0$ satisfying (13). We may assume that $(z_n)$ converges to $\hat{z} \in \gamma_0$, and we have, by (13),

\[ f^n(\hat{z}) = \lim_{n \to \infty} f^n(z_n) \in \gamma_m \quad \text{for each} \quad m \geq 0. \tag{14} \]

Using (12) we get $\hat{z} \in I(f)$ and hence $I(f)$ is non-empty. This proves the first assertion of Theorem 1.1.

The second assertion will be established by modifying the method of Rippon and Stallard [16], so assume that $f$ is $K$-quasiregular in $\mathbb{R}^3$ and take $\hat{z}$ satisfying (14). As before let $A_n = G^n_*$ be the component of $\mathbb{R}^3 \setminus G_n$ containing $\infty$, and let $L_n$ be the component of $f^{-n}(A_n)$ containing $\hat{z}$, which is well-defined since $f^n(\hat{z}) \in \gamma_n$ and $\gamma_n = \partial A_n$ by definition.

Lemma 5.3. $L_n$ is closed and unbounded.

Proof. $L_n$ is closed since $A_n$ is closed, and $L_n$ is unbounded by Lemma 3.2.

Lemma 5.4. We have $L_{n+1} \subseteq L_n$ for $n = 0, 1, \ldots$.

Proof. Suppose that $f^{n+1}(z') \in A_{n+1}$ but $f^n(z') \notin A_n$. Thus either $|f^n(z')| \leq R$ or $f^n(z') \in G_n$, from which we obtain $f^{n+1}(z') \notin A_{n+1}$, in the first case from (6) and (7) and in the second case from (8), and this is a contradiction. Hence if $z' \in L_{n+1}$ then $z'$ lies in a component of $f^{-n-1}(A_{n+1})$ which contains $\hat{z}$, and this component in turn lies in a component of $f^{-n}(A_n)$. Hence we get $z' \in L_n$.

We may now write

\[ K_n = L_n \cup \{\infty\}, \quad \{\hat{z}, \infty\} \subseteq K_{n+1} \subseteq K_n, \quad \{\hat{z}, \infty\} \subseteq K = \bigcap_{n=0}^{\infty} K_n. \]

Since $K_n$ is compact and connected so is $K$ [11, Theorem 5.3, p.81]. Let $\Gamma$ be the component of $K \setminus \{\infty\}$ which contains $\hat{z}$. Then $\Gamma$ is unbounded by Lemma 3.1. Now for $w \in \Gamma$ we have $w \in L_n$ and so $f^n(w) \in A_n = G^*_n$, so that $w \in I(f)$ by (7). This completes the proof of Theorem 1.1.

We do not know whether the second conclusion of Theorem 1.1 holds if $f$ is only quasiregular on the set $B_R$ in (1), but this seems unlikely. The difficulty is that for large $n$ we cannot control the behaviour of $f^n$ near $S(0, R)$ and so the component $L_n$ in Lemma 5.3 may in principle be bounded.

6. A Quasiregular mapping $f$ for which $I(f)$ has a bounded component

To show that there exists a quasiregular mapping $f : \mathbb{C} \to \mathbb{C}$ such that the closure of the escaping set $I(f)$ has a bounded component, we begin by constructing a quasiconformal map $g$ with the following properties. For each $z$ in the punctured disc $A := \{z \in \mathbb{C} : 0 < |z| < 1\}$ the iterates $g^n$ satisfy $\lim_{n \to \infty} |g^n(z)| = 1$, and we have $\lim_{n \to \infty} g^n(1/2) = 1$. On the other hand there exist annuli $A_n \subseteq A$ such that $g$ maps $A_n$ onto $A_{n+1}$, but with sufficient rotation that for each $z \in A_n$ infinitely many of the forward images $g^n(z)$ lie away from 1. A map $h$ is then obtained from $g$ by conjugation with a Möbius map $L$ which sends 1 to $\infty$, and finally $h$
is interpolated on a sector to ensure that the resulting function has an essential singularity at infinity.

We will use the fact that if \( p \) is quasiregular on a domain \( D \subseteq \mathbb{C} \) and

\[
p_z = \frac{\partial p}{\partial z} = \frac{1}{2} \left( \frac{\partial p}{\partial x} - i \frac{\partial p}{\partial y} \right)\]

is bounded below in modulus on \( D \), and if \( q \) is continuous and such that the partial derivatives \( q_x, q_y \) are sufficiently small on \( D \), then \( p + q \) is quasiregular on \( D \). If \( 0 \not\in D \cup p(D) \) the same property may be applied locally to \( \log p \) as a function of \( \log z \).

Turning to the detailed construction, we define \( a : [1, 2] \rightarrow [0, \pi/4] \) by

\[
a(r) = \frac{\pi}{4} - \arcsin \left( \frac{\sqrt{2}}{2r} \right).
\]

Then an application of the sine rule shows that the line segment \( \text{Re } z = 1 + \text{Im } z, \quad 1 \leq |z| \leq 2 \), is parametrized by \( z = re^{ia(r)} \).

For \( c > 0 \) we define \( g : \mathbb{C} \rightarrow \mathbb{C} \) as follows. Let \( g(0) = 0 \) and for \( z = re^{it} \) with \( r > 0 \) and \( -\pi \leq t \leq \pi \) set:

\[
g(z) = \begin{cases} 
\frac{4}{\pi} r \exp \left( i \left( t + c |\sin t| \right) \right), & 0 < r < \frac{1}{2}; \\
\frac{1}{\pi - r} \exp \left( i \left( t + c |\sin t| + c(1 - r)^2 |\sin \left( \frac{\pi}{1 - r} \right) \right) \right), & \frac{1}{2} \leq r < 1; \\
r \exp \left( i \left( t + c(2 - r) \sin \left( \frac{|t| - a(r)}{\pi - a(r)} \pi \right) \right) \right), & 1 \leq r \leq 2, |t| < a(r); \\
r \exp(it), & 1 \leq r \leq 2, |t| \geq a(r); \\
r \exp(it), & r > 2.
\end{cases}
\]

Then \( g \) is continuous on \( \mathbb{C} \). Moreover, if \( c \) is sufficiently small then \( g \) is quasiconformal, and in particular we choose \( c < \pi/4 \). Note that, by the choice of \( a(r) \),

(15) \[ g(z) = z \quad \text{if} \quad \text{Re } z \geq |\text{Im } z| + 1. \]

For \( n \in \mathbb{N} \) we have

(16) \[ g \left( 1 - \frac{1}{n + 1} \right) = 1 - \frac{1}{n + 2}. \]

For \( n \in \mathbb{N}, n \geq 2 \), we consider the annulus

\[
A_n := \left\{ z \in \mathbb{C} : 1 - \frac{1}{n + 1/4} < |z| < 1 - \frac{1}{n + 3/4} \right\}.
\]

Then \( g(A_n) = A_{n+1} \).

**Lemma 6.1.** For each \( z \in A_n \) with \( \text{Re } z > 0 \) there exists \( k \in \mathbb{N} \) with \( \text{Re } g^k(z) \leq 0 \).

**Proof.** Let \( z \in A_n \), and suppose first that \( 0 < t := \arg z < \pi/2 \). Then

(17) \[ \pi > t + \frac{\pi}{2} > t + 2c \geq \arg g(z) \geq t + c \sin t \geq t + \frac{2c}{\pi} t = \left( 1 + \frac{2c}{\pi} \right) t. \]
On the other hand if $-\pi/2 < t = \arg z \leq 0$ then
\begin{equation}
\frac{c}{2} > \arg g(z) \geq t + c |\sin t| + \frac{c\sqrt{2}}{2(m + 3/4)^2} \geq \left(1 - \frac{2c}{\pi}\right) t + \frac{c'}{(m + 1)^2} > -\frac{\pi}{2}
\end{equation}
where $c' := \frac{1}{2}c\sqrt{2}$. In particular, (17) and (18) both hold with $\arg g(z)$ the principal argument.

Suppose then that there exists $z \in A_n$ with $\Re g^k(z) > 0$ for all integers $k \geq 0$, and set $t_k = \arg g^k(z) \in (-\pi/2, \pi/2)$. Then $g^k(z) \in A_{n+k}$. If there exists $k \geq 0$ with $0 < t_k < \pi/2$ then by repeated application of (17) we obtain $k' > k$ with $t_{k'} \in (\pi/2, \pi)$, a contradiction. Hence we must have $-\pi/2 < t_k \leq 0$ for all $k \geq 0$. But then repeated application of (18) gives, for large $k$,
\begin{equation}
t_{k-1} \geq \left(1 - \frac{2c}{\pi}\right)^{k-1} t_0, \quad t_k \geq \left(1 - \frac{2c}{\pi}\right) t_{k-1} + \frac{c'}{(n+k)^2} > 0,
\end{equation}
again a contradiction. \hfill \Box

With the Möbius transformation
\begin{equation}
L(z) = \frac{1}{1 - z}
\end{equation}
we now consider the map $h := L \circ g \circ L^{-1}$. Then $h$ is a quasiconformal self-map of the plane. Moreover, (15) gives $h(z) = z$ if $\Re L^{-1}(z) \geq |\Im L^{-1}(z)| + 1$, which is equivalent to $\Re z \leq -|\Im z|$, and we have
\begin{equation}
L(A_n) \subseteq \{z \in \mathbb{C} : \Re z > 0\} \quad \text{and} \quad h(L(A_n)) = L(A_{n+1}),
\end{equation}
using the fact that $g(A_n) = A_{n+1}$.

It follows from (16) that
\begin{equation}
h(n+1) = n + 2 \quad \text{for} \quad n \in \mathbb{N},
\end{equation}
and we deduce at once that $2 \in I(h)$. Next we show that $L(A_n) \cap I(h) = \emptyset$ for every integer $n \geq 2$. In fact, suppose that $n \geq 2$ and $w \in L(A_n) \cap I(h)$. Then there exists $j_0 \in \mathbb{N}$ such that $|h^j(w)| > 1$ for $j \geq j_0$. Put $w := h^{j_0}(u)$ and $m := n + j_0$. Then $L^{-1}(w) \in A_m$ by (19), and Lemma 6.1 gives $k \geq 0$ with $\Re g^k(L^{-1}(w)) \leq 0$. Since $|L(z)| \leq 1$ for $\Re z \leq 0$ we deduce that
\begin{equation}
|h^{k+j_0}(w)| = |h^k(w)| = |L(g^k(L^{-1}(w)))| \leq 1,
\end{equation}
contradicting the choice of $j_0$. Thus $L(A_n) \cap I(h) = \emptyset$.

Since $A_2$ separates $\frac{1}{2}$ from 1 it follows that 2 lies in the bounded component of the complement of $L(A_2)$, and we deduce that the component of $\overline{I(h)}$ containing 2 is bounded.

To construct a quasiregular map $f : \mathbb{C} \to \mathbb{C}$ with an essential singularity at $\infty$ for which the closure of $I(f)$ has a bounded component we put $f(z) = h(z)$ for $\Re z \geq -|\Im z|$ and $f(z) = z + d \exp(z^4)$ for $\Re z \leq -|\Im z| - 1$, where $d$ is a small positive constant. In the remaining region $\Omega$ we define $f$ by interpolation, using $f(z) = z - d\phi(z)$, $\phi(z) = (\Re z + |\Im z|) \exp(z^4)$ for $-1 < \Re z + |\Im z| < 0$.

Since $\exp(z^4)$ tends to 0 rapidly as $z$ tends to infinity in $\Omega$, it is then clear that the partial derivatives of $\phi$ are bounded on $\Omega$, so that $f$ is quasiregular on $\Omega$ because $d$ is small.
In particular we have $f(z) = h(z)$ for $\text{Re} z > 0$ and so it follows from (20) that $2 \in I(f)$, whereas $L(A_n) \cap I(f)$ is again empty using (19). Thus the component of $\overline{I(f)}$ containing 2 is bounded.

7. The quasimeromorphic case

Let $f$ be non-constant and K-quasimeromorphic on the set $B_R$ defined in (1), with a sequence of poles tending to $\infty$, and set $R_{-1} = R$. Choose $x_j, D_j, R_j$ for $j = 0, 1, 2, \ldots$ as follows. Each $x_j$ is a pole of $f$, and $D_j$ is a bounded component of the set $\{x \in B_R : R_j < |f(x)| \leq \infty\}$ which contains $x_j$ but no other pole of $f$, such that $D_j$ is mapped by $f$ onto $\{y \in \mathbb{R}^N : R_j < |y| \leq \infty\}$. Moreover, by choosing $R_{j+1}$ and $x_{j+1}$ sufficiently large, we may ensure that

$$|x_{j+1}| > 4R_j \quad \text{and} \quad D_{j+1} \subseteq \{x \in \mathbb{R}^N : 2R_j < |x| < \infty\} \quad \text{for} \quad j \geq -1.$$  \hspace{1cm} (21)

Since $|f(x)| = R_j$ for all $x \in \partial D_j$ we may write, for $j \geq 0$, using (21),

$$C_j = \{x \in D_j : f(x) \in D_{j+1}\} \subseteq \overline{C_j} \subseteq D_j.$$  \hspace{1cm} (22)

Now set

$$X_0 = \overline{C_0}, \quad X_{j+1} = \{x \in X_j : f^{j+1}(x) \in \overline{C_{j+1}}\}. \hspace{1cm} (23)$$

Evidently $X_0$ is compact. Assuming that $X_j$ is compact, it then follows that $X_{j+1}$ is the intersection of a compact set with the closed set $f^{-1}(\overline{C_{j+1}})$ and so is compact. Hence the $X_j$ form a nested sequence of compact sets. We assert that

$$f^j(X_j) = \overline{C_j}. \hspace{1cm} (24)$$

We clearly have $f^j(X_j) \subseteq \overline{C_j}$ by (23), and (24) is obviously true for $j = 0$, so assume the assertion for some $j \geq 0$ and take $w \in C_{j+1}$. Since $f$ maps $D_j$ onto $\{y \in \mathbb{R}^N : R_j < |y| \leq \infty\}$, it follows from (21) and (22) that there exists $v \in C_j$ with $f(v) = w$. Hence there exists $x \in X_j$ with $f^j(x) = v$ and $f^{j+1}(x) = w$, completing the induction.

Again since $f$ maps $D_j$ onto $\{y \in \mathbb{R}^N : R_j < |y| \leq \infty\}$, we evidently have $C_j \neq \emptyset$ and so $X_j$ is non-empty by (24). Hence there exists $x$ lying in the intersection of the $X_j$, so that $f^j(x) \in \overline{C_j}$ and $x \in I(f)$ by (21) and (22).

8. Proof of Lemma 3.2

To establish Lemma 3.2 let $E$ and $g$ be as in the statement and assume that $g^{-1}(E)$ is non-empty since otherwise there is nothing to prove. Note first that $g^{-1}(E)$ is a closed subset of $\mathbb{R}^N$ by continuity. Thus

$$F = g^{-1}(E) \cup \{\infty\}$$

is a compact subset of $\mathbb{R}^N$. In order to prove Lemma 3.2 it therefore suffices in view of Lemma 3.1 to show that $F$ is connected. Suppose that this is not the case. Then there is a partition of $F$ into non-empty disjoint relatively closed (and so closed) sets $H_1, H_2$ such that $\infty \in H_2$. Let $W = \mathbb{R}^N \setminus H_2$. Then $W$ is an open subset of $\mathbb{R}^N$, and $g(W \setminus H_1) \cap E = \emptyset$. Moreover, $H_1$ is a closed subset of $\mathbb{R}^N$ and so compact, and hence a compact subset of $\mathbb{R}^N$ since $\infty \in H_2$. Thus $g(H_1)$ is compact and so a non-empty closed subset of $E$.

Now suppose that there exist $y_n \in E \setminus g(H_1)$ with $y_n \to \tilde{y} \not\in E \setminus g(H_1)$. Since $E$ is compact we have $\tilde{y} \in E$ and so $\tilde{y} \in g(H_1)$. Hence there exists $\tilde{x} \in H_1$ with $g(\tilde{x}) = \tilde{y}$.
and for large enough $n$ there exists $x_n$ close to $\tilde{x}$ with $g(x_n) = y_n \in E \setminus g(H_1)$. But then we must have $x_n \in H_1$, since $g(W \setminus H_1) \cap E = \emptyset$, and this is a contradiction. So $E \setminus g(H_1)$ is also closed, but evidently non-empty since $g(\mathbb{R}^N) \subseteq \mathbb{R}^N$ and $\infty \in E$, which contradicts the hypothesis that $E$ is connected.

References