Math 420 *Algebra* - Exam II
October 29, 2004

Instructions: You have **50 min** to complete the exam (**5 problems**). This is a closed book, closed notes exam. Use of calculators is not permitted. Show all your work for full credit. **Write your name on the blue book cover.**

Print your name : ______________________________

Problem	Max Points	Your Score	Problem	Max Points	Your Score
1	10		4	15	
2	15		5	20	
3	20				
			Total	80	

(1) Find the multiplicative inverse of [351] in $\mathbb{Z}_{6669}$.

Solution $6669 = 351 \cdot 19$. Hence $(351, 6669) \neq 1$ and $[351]_{6669}$ has no multiplicative inverse in $\mathbb{Z}_{6669}$.

(2) Let $(a, n) = 1$. The smallest positive integer k such that $a^k \equiv 1 \pmod{n}$ is called the *multiplicative order of* $[a]$ in $\mathbb{Z}_n^\times$.

(a) Show that $k \mid \varphi(n)$.

(b) How many elements does $\mathbb{Z}_{16}^\times$ have?

(c) Compute the multiplicative order of [5] in $\mathbb{Z}_{16}^\times$.

(3) Consider $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2 + z + 1$.

(a) Determine if f is one-to-one.

(b) Determine if f is onto. If yes, find $g : \mathbb{C} \to \mathbb{C}$ such that $f \circ g = 1_{\mathbb{C}}$. (Recall that a function $f : A \to B$ is onto if and only if there exists $g : B \to A$ such that $f \circ g = 1_B$.)

Solution (a) f is not one-to-one since $f(0) = f(-1) = 1$.

(b) Given $w \in \mathbb{C}$, $f(z) = z^2 + z + 1 = w$ has two solutions

$$\frac{-1 \pm \sqrt{1 - 4(1 - w)}}{2} \in C.$$

Therefore, f is onto. $g(w) = \frac{-1+\sqrt{1-4(1-w)}}{2}$ (or $g(w) = \frac{-1-\sqrt{1-4(1-w)}}{2}$) satisfies $f(g(w)) = w$, hence $f \circ g = 1_{\mathbb{C}}$.

(4) Let $\mathcal{P}$ be the collection of all horizontal lines in $\mathbb{R}^2$.

(a) Show that $\mathcal{P}$ defines a partition.

(b) Recall that a partition naturally induces an equivalence relation. What is the equivalence relation $\sim$ corresponding to $\mathcal{P}$?

(c) Give a *geometric description* of the set $\mathbb{R}^2/\sim$. (**Hint:** Each horizontal line collapses to a point in $\mathbb{R}^2/\sim$.)

Solution (a) Every point $(a, b) \in \mathbb{R}$ belongs to exactly one horizontal line $y = b$. Hence $\mathcal{P}$ is a partition by definition.

(b) $(a_1, b_1) \sim (a_2, b_2)$ if $b_1 = b_2$.

(c) There is a natural one-to-one onto map

$$\begin{array}{rcl} \mathbb{R}^2/\sim & \to & y-\text{axis} \\ [(a,b)] & \mapsto & (0,b). \end{array}$$

Hence $\mathbb{R}^2/\sim$ is simply $\mathbb{R}$.

(5) Consider the following permutations in S_7:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}.$$

(a) Compute $\sigma\tau\sigma^{-1}$.

(b) Write $\sigma\tau\sigma^{-1}$ as a product of disjoint cycles.

(c) Compute the order of $\sigma\tau\sigma^{-1}$.

(d) Write $\sigma\tau\sigma^{-1}$ as a product of transpositions.