MATH 550 TOPOLOGY HOMEWORK

DUE APRIL 22, 2005

(1) Let (Y, c) be a pointed topological space such that Y is locally path connected and simply connected, and let $p : (E, e) \to (X, b)$ be a covering map.

(a) Show that every map $f : (Y, c) \to (X, b)$ can be uniquely lifted to a map $g : (Y, c) \to (E, e)$.

Solution. The uniqueness follows from the uniqueness of a lift to a covering space from a connected space. It remains to show the existence of a lift $g : (Y, c) \to (E, e)$. Define $g(c) = e$, and let $y \in Y$. Since p is surjective and E is path-connected, X is also path-connected. Let γ be a path joining b to $x := f(y)$ and $\widetilde{\gamma}$ be the unique lift to E starting at e. We define g by

$$g(y) = \widetilde{\gamma}(1).$$

(i) g is well defined: Let σ be another path joining b to x and $\widetilde{\sigma}$ be its lift. We have

$$\begin{aligned} \gamma &= (\gamma \cdot \sigma^{-1}) \cdot \sigma \\ \widetilde{\gamma} &= \widetilde{\gamma \cdot \sigma^{-1}} \cdot \widetilde{\sigma} \simeq_p \widetilde{\sigma} \\ \therefore\ \widetilde{\sigma}(1) &= \widetilde{\gamma}(1). \end{aligned}$$

(ii) g is continuous : let $z \in E$ and $x = p(z)$. Let U be a neighborhood of x that is evenly covered by p and V denote the component of $p^{-1}(U)$ that contains z. Since Y is locally path-connected, there is a path-connected neighborhood N of a point $y \in p^{-1}(x)$.

Let $n \in N$ and α be a path joining y to n. If γ is a path joining b to x, then $\gamma \cdot p \circ \alpha$ is a path joining b to $f(n)$. Moreover, $p \circ \alpha$ is in U, hence $\widetilde{\alpha}$ starting at z is contained in V. Therefore, $g(n) \in V$ and $g(N) \subset V$.

□

(b) Suppose in addition that X is locally path connected and E is simply connected. Show that if f is a covering map, then the unique lift g is a homeomorphism.

Solution. Under the assumptions, E is locally path-connected and simply connected. Hence by (a), there is a map $h : E \to Y$ such that $p = f \circ h$. By the uniqueness of the lifting, $g \circ h = 1_Y$ and $h \circ g = 1_E$. Hence h is a homeomorphism.

$\square$

(2) Show that any two-sheeted covering admits a unique G-covering structure.

Solution. Let $p : E \to X$ be a two-sheeted covering and let $G = \mathbb{Z}/2\mathbb{Z} = \langle -1 \rangle$ act on E by interchanging the sheets: Define the action $\sigma : G \times E \to E$ by $\sigma(-1, e) = -e$ where e and $-e$ are the two points over $p(e)$. Note that σ is a continuous map since it simply interchanges the two sheets, mapping one open set V to another open set $-V = \{-e : e \in V\}$ that is isomorphic to V (verify this). The condition $\sigma(g, \sigma(h, e)) = \sigma(g \cdot h, e)$ doesn't quite require a proof, since G has only -1 and 1. Lastly, X is the orbit space E/G since $p^{-1}(x) = \{e, -e\} = G \cdot e$ is a G-orbit for every x and the topology on X is the quotient topology. By a previous lemma p can be identified with the quotient projection map. Uniqueness is obvious: $\sigma(-1, e)$ can only be either e or $-e$. $\square$

(3) Show that S^n is simply connected for $n \geq 2$.

Solution. Let $\gamma : I \to S^n$ be a loop based at x. Since $n \geq 2$, $\gamma(I) \neq S^n$. We can assume that $y := (1, 0, \ldots, 0) \in S^n \setminus \gamma(I)$. Since $\gamma(I)$ is compact and S^n is Hausdorff, it is a closed subset of S^n. Hence there is $r > 0$ such that $\overline{B}_r(y) = \{z \in S^n \mid \|z - y\| \leq r\}$ is contained in $S^n \setminus \gamma(I)$. The assertion follows from that $\gamma(I)$ is contained in $S^n \setminus \overline{B}_r(y)$ which is homeomorphic to $S^n \setminus \overline{B}_1(y)$, which in turn is homeomorphic to a contractible space $D^n = \{x \in R^n \mid \|x\| < 1\}$ by the map $z = (z_0, \ldots, z_n) \mapsto (z_1, \ldots, z_n)$. (Verify that this is a homeomorphism!) $\square$

(4) Let X be the figure-eight and $p : E \to X$ be the three-sheeted covering introduced in class. What is the group $\mathrm{Aut}(E/X)$ of covering transformations?

Solution. Let $f \in \mathrm{Aut}(E/X)$. Let q denote the intersection of the two loops of X, and q_1, q_2, q_3 denote the three points over q, ordered from right to left. Then f maps q_1 to one of these three points. But $f(q_1)$ cannot be mapped to q_2 or q_3 since the loop A on the right of q_1 must be mapped to a loop over A. Hence $f(q_1) = q_1$. Since E is connected and 1_E and f agrees at one point, $f \equiv 1_E$. $\square$

(5) Let G be the subgroup of the group of homeomorphisms of the plane to itself generated by $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (-x, y + 1)$.

(a) Show that the G action on $\mathbb{R}^2$ is properly discontinuous.

Solution. For any point $q \in \mathbb{R}^2$, let B_q denote the ball of radius $1/4$ centered at q. If $g \cdot q \neq q$, then $g \cdot q = q + (m, n)$ for some $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Hence $\|g \cdot q - q\| \geq 1$ if $g \cdot q \neq q$. Since $g \cdot q \neq q$ for any $q \in \mathbb{R}^2$ and $g \neq e \in G$, it follows that $g \cdot B_p \cap h \cdot B_p = \emptyset$ for any $g \neq h \in G$. □

(b) What is the quotient space $\mathbb{R}^2/G$?

Solution. The quotient space is the Klein bottle. The covering described in (a) is the universal covering, and we conclude that the fundamental group of the Klein bottle is

$$\langle \alpha, \beta \,|\, \alpha\beta = \beta\alpha^{-1} \rangle.$$

□