[1]

MATH 550 TOPOLOGY HOMEWORK

DUE 12PM ON MAY 8, 2005

(1) A covering map $p : E \to X$ is said to be *regular* if $p_* \pi_1(E, e)$ is a normal subgroup of $\pi_1(X, x)$. Show that a G-covering $p : E \to X$ is regular if E is connected and locally path-connected.

Solution. We isolate the key element of the proof as a lemma:

Lemma 1. *Suppose that $p : (E, e) \to (X, x)$ is a G-covering and τ is a loop based at e. For any $e' \in p^{-1}(x)$, the unique lift of $p \circ \tau$ starting at e' is a loop.*

Proof. Since X is the quotient of E by G, $e' = g.e$ for some $g \in G$. Let $\tau' := g.\tau$. It starts at $\tau'(0) = g.\tau(0) = g.e = e'$ and $p \circ (g.\tau) = p \circ \tau$. Hence it is the unique lift of $p \circ \tau$ starting at e'. It is a loop since $\tau'(1) = g.\tau(1) = g.e = e'$. □

Back to the solution of the problem in hand. Let τ be a loop in E based at e and $[\sigma] \in \pi_1(X, x)$. We want to show that $[\sigma][p \circ \tau][\sigma^{-1}] = [p \circ \rho]$ for some $[\rho] \in \pi_1(E, e)$. Let $\widetilde{\sigma}$ be the unique lift of σ starting at e. Then $\widetilde{\sigma}^{-1}$ is the unique lift of σ^{-1} starting at $\widetilde{\sigma}(1)$. By the lemma, the unique lift τ' of $p \circ \tau$ starting at $\widetilde{\sigma}(1)$ is a loop based at $\widetilde{\sigma}(1)$. Let $\rho := \widetilde{\sigma}.\tau'.\widetilde{\sigma}^{-1}$. It is a loop based at e and

$$\begin{aligned} p \circ \rho &= (p \circ \widetilde{\sigma}).(p \circ \tau').(p \circ \widetilde{\sigma}^{-1}) \\ &= \sigma.(p \circ \tau).\sigma^{-1}. \end{aligned}$$

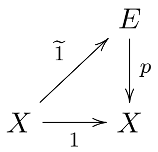
Remark 1. Note that Lemma 1 does not hold in general if p is not a G-covering: Let X be the figure eight and $p : E \to X$ be the three-sheeted covering introduced in class. Retaining the notation from the solution of the previous homework, the unique lift of the loop A starting at q_1 is a loop but the one starting at q_2 or q_3 is not a loop.

□

(2) Show that a simply connected and locally path-connected space has only trivial coverings.

[1]Solution will be posted on May 8 at 5PM.

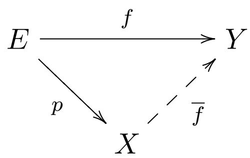
Solution. Let X be a simply connected and locally path-connected space and $p : E \to X$ be a covering map. Since the identity map $1 : X \to X$ is a simply connected cover, there is a unique lift $\widetilde{1} : X \to E$ such that $p \circ \widetilde{1} = 1$:



Clearly $\widetilde{1}$ is injective. Let $x \in X$ and N be a path-connected neighborhood of x that is evenly covered by p, say $p^{-1}(N) = \amalg V_\alpha$. If V is the component of $p^{-1}(N)$ containing $\widetilde{1}(x)$, then $\widetilde{1}(N) \subset V$ since N is path-connected. It follows that $\widetilde{1}$ is a local homeomorphism mapping N homeomorphically onto V. Also, if $y \in p^{-1}(N)$ and $y \notin V$ then $y \in V_\alpha \neq V$ and $V_\alpha \subset E \setminus \widetilde{1}(X)$. This means that $E \setminus \widetilde{1}(X)$ is open. Hence $\widetilde{1}(X)$ is a component of E and we are done.

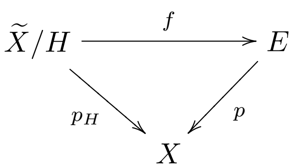
□

(3) Suppose that a topological group G acts on E properly discontinuously, and let $p : E \to X := E/G$ be the canonical projection map to the orbit space. Prove that any G-invariant continuous map $E \to Y$ factors through X. i.e. for any map $f : E \to Y$ such that $f(g \cdot e) = f(e)$ for all $g \in G$ and $e \in E$, there is a map $\overline{f} : X \to Y$ such that $\overline{f} \circ p = f$.

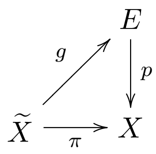


Solution. Define $\overline{f}$ by $\overline{f}(x) = f(e)$ where e is any point of $p^{-1}(x)$. It is well-defined because of G-invariance of f. It is continuous since X has the quotient topology: Let $U \subset Y$ be an open subset. Then $p^{-1}\left(\overline{f}^{-1}(U)\right) = f^{-1}(U)$ is open hence by definition of quotient topology $\overline{f}^{-1}(U)$ is open in X. □

(4) Let X be a topological space and $\pi : \widetilde{X} \to X$ be a universal covering of X. Let $H \subset \pi_1(X)$ be a subgroup and $p : E \to X$ be a connected covering map such that $p_*\pi_1(E) = H$. Show that there is a canonical homeomorphism $f : \widetilde{X}/H \to E$ such that the following diagram is commutative:



Solution. By the universal property of $\pi : \widetilde{X} \to X$, there is a unique lift $g : \widetilde{X} \to E$ such that $p \circ g = \pi$:



Since π and p are covering maps, g is also a covering map. Indeed, it is a universal covering of E. By the uniqueness of universal covering, $E = \widetilde{X}/H$ and $g : \widetilde{X} \to E$ can be identified with $\widetilde{X} \to \widetilde{X}/H$. The above diagram determines $p : E = \widetilde{X}/H \to X$ completely as $p([x]) = \pi(x) = p_H([x])$. $\square$