

Not Real Numbers

This is an extra credit “project” which may help you sort out what makes a sequence Cauchy (as opposed to convergent) and what the real numbers really are. The idea is to start with the rational numbers and a different notion of “absolute value,” and see how the process of creating the real numbers would change. I’ve labelled what you need to do as exercises. The due date on this is the midterm exam date.

Every non-zero integer n can be written uniquely as a power of 2 times an odd number: $n = 2^k m$, where $k \geq 0$ and m is odd. Call that exponent k the “2-order” of n and write $k = \text{ord}_2(n)$. Some examples are: $\text{ord}_2(5) = 0$, $\text{ord}_2(10) = 1$, $\text{ord}_2(24) = 3$ and $\text{ord}_2(123432421) = 0$. For a rational number p/q , where p and q are integers, we define $\text{ord}_2(p/q) = \text{ord}_2(p) - \text{ord}_2(q)$. Some examples are $\text{ord}_2(37/5) = 0$, $\text{ord}_2(36/5) = 2$ and $\text{ord}_2(15/18) = -1$. Notice that it doesn’t matter how you write the rational number (e.g., $\text{ord}_2(36/5) = \text{ord}_2(72/10)$).

Definition: The 2-adic absolute value of a non-zero rational number p/q is defined to be $2^{-\text{ord}_2(p/q)}$, and we write $|p/q|_2$ to denote this 2-adic absolute value. We define $|0|_2 = 0$.

Exercise 1: a) Find $|324/17|_2$, $|325/324|_2$ and $|256|_2$.

b) Show that $|a \cdot b|_2 = |a|_2 \cdot |b|_2$ for any two rational numbers a and b .

Suppose n and m are integers, and write $n = 2^k a$ and $m = 2^l b$ where a and b are odd integers. If $k \leq l$, then $n + m = 2^k(a + 2^{l-k}b)$. In particular, $\text{ord}_2(n + m)$ is at least as large as $\text{ord}_2(n) = \min\{\text{ord}_2(n), \text{ord}_2(m)\}$. If $k < l$, then $a + 2^{l-k}b$ will be odd, so that we even have $\text{ord}(n + m) = \text{ord}(n)$. If $k = l$, though, $a + 2^{l-k}b$ will be even, so that $\text{ord}(n + m)$ can be larger than $\text{ord}_2(n) = \text{ord}_2(m)$.

Now suppose a and b are non-zero rational numbers, and write $a = p/q$ and $b = r/s$ where p, q, r, s are integers. Then $a = ps/qs$ and $b = rq/qs$. We have $|a + b|_2 = |ps + rq|_2/|qs|_2$. By the remarks above, $\text{ord}_2(ps + rq) \geq \min\{\text{ord}(ps), \text{ord}(rq)\}$, which means that $|ps + rq|_2 \leq \max\{|ps|_2, |rq|_2\}$. Dividing through by $|qs|_2$, you get

$$(*) \quad |a + b|_2 \leq \max\{|a|_2, |b|_2\},$$

with equality if $|a|_2 \neq |b|_2$.

What all this means is that the 2-adic absolute value is just that - an absolute value:

i) $|a|_2 \geq 0$, with equality if and only if $a = 0$,

ii) $|a \cdot b|_2 = |a|_2 \cdot |b|_2$, and

iii) $|a + b|_2 \leq |a|_2 + |b|_2$.

Of course, (*) is stronger than the usual triangle inequality iii here, and that leads to strange and interesting stuff.

Exercise 2: a) Show that the sequence $2, 4, 8, 16, \dots$ is a Cauchy sequence using the 2-adic absolute value. Does it converge (in \mathbb{Q})?

b) Give an example of a sequence that is Cauchy using the usual absolute value, but not Cauchy using the 2-adic absolute value.

For two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers (where we use the 2-adic absolute value to define Cauchy), say $\{a_n\}$ is equivalent to $\{b_n\}$, and write $\{a_n\} \sim \{b_n\}$, if the sequence $\{a_n - b_n\} \rightarrow 0$. Note that this now means that, given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - b_n|_2 < \epsilon$ for all $n \geq N$.

Lemma: This is an equivalence relation.

Proof: The exact same proof we had for the real numbers works word for word, just changing the absolute values there with 2-adic ones!

Exercise 3: Show that the Cauchy sequence $2, 4, 8, 16, \dots$ is equivalent to the sequence of all 0's: $0, 0, 0, \dots$

Definition: The 2-adic numbers \mathbb{Q}_2 is the set of equivalence classes of Cauchy sequences of rational numbers, where we use the 2-adic absolute value to define Cauchy.

Past this point, almost everything we did with real numbers works with 2-adic numbers. The definitions and proofs are practically identical; you just need to replace the absolute values with 2-adic ones. The only real (pun unintended) difference is that we don't talk about inequalities with \mathbb{Q}_2 (there is no notion of " $a < b$ " for 2-adic numbers). But what we get out sure isn't the real numbers! Is it obvious, for example, that there is (or isn't!) a square root of 2 in \mathbb{Q}_2 ? In other words, is there a Cauchy sequence (using 2-adic absolute values) $\{a_n\}$ of rational numbers where the a_n^2 's get "close" to 2? Here "close" would mean that $|a_n^2 - 2|_2 < \epsilon$ for a given positive ϵ .

Exercise 4: a) Suppose that a is a rational number and $\epsilon < 1/2$. If $|a - 2|_2 < \epsilon$, show that $|a|_2 = 1/2$. (You may use the strong form of the triangle inequality (*) if you wish.)

b) If $\epsilon < 1/2$, is there any rational number b with $|b^2 - 2|_2 < \epsilon$? Is there a square root of 2 in \mathbb{Q}_2 ?