The Degree Function for Polynomials

Suppose \( P(X) = a_nX^n + \cdots + a_0 \in \mathbb{Q}[X] \) and \( a_n \neq 0 \). Then the degree of \( P(X) \) is \( n \). The degree of 0 is defined to be \(-\infty\). We write \( \deg(P(X)) \) (or just \( \deg(P) \)) for the degree of the polynomial \( P(X) \).

Aside: for our purposes, we’ll say \(-\infty < n \) and \(-\infty + n = -\infty \) for all integers \( n \).

Lemma: For any two polynomials \( P(X) \) and \( Q(X) \),

\[
\deg(P \times Q) = \deg(P) + \deg(Q)
\]

and

\[
\deg(P + Q) \leq \max\{\deg(P), \deg(Q)\}.
\]

Lemma: If \( P(X) \times Q(X) = 0 \), then either \( P(X) = 0 \) or \( Q(X) = 0 \).

Division Algorithm: For any polynomials \( A(X) \) and \( B(X) \) with \( B(X) \neq 0 \), there are polynomials \( Q(X) \) and \( R(X) \) with

\[
A(X) = Q(X) \times B(X) + R(X)
\]

and \( \deg(R) < \deg(B) \).

We can prove this in a manner similar to the proof in the text for the division algorithm (Theorem 1.1.3).
**Proof:** If $A(X) = 0$, then we simply let $Q(X) = R(X) = 0$, too, and we’re done.

Suppose $A(X) \neq 0$. Consider the set of all polynomials of the form $A(X) - Q(X)B(X)$. If 0 is in this set, then we’re done. If not, then this set has an element of least degree (since the set of degrees of such polynomials is a non-empty subset of the natural numbers). Let $R(X)$ be such a polynomial. Suppose $\deg(R) \geq \deg(B)$. We can write

$$R(X) = r_mX^m + \cdots + r_0,$$

where $r_m \neq 0$. Write

$$B(X) = b_nX^n + \cdots + b_0,$$

where $b_n \neq 0$ and $n \leq m$. Then

$$R(X) - \frac{r_m}{b_n}X^{m-n}B(X) = 0X^m + \cdots$$

is also in the set of polynomials above, and moreover its degree is less than $m$. This contradicts the way $R(X)$ was chosen, so $\deg(R) < \deg(B)$. 