13. Suppose $G$ is a group and $g \in G$ satisfies $g^2 = g$, i.e., $g \cdot g = g$. Letting $e \in G$ denote the identity element, we have $g \cdot g = g \cdot e$, and by left cancellation (Proposition 3.1.6) $g = e$.

19. Suppose $G$ is a group such that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Let $a, b \in G$. Then by Proposition 3.1.2 (one of our “little results”) $(ba)^{-1} = a^{-1}b^{-1}$. We thus have $(ba)^{-1} = (ab)^{-1}$, and taking inverses of both sides (using another “little result”) gives $ba = ab$. Thus, $G$ is abelian.

Now suppose $G$ is abelian and $a, b \in G$. Then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$ by Proposition 3.1.2 and the hypothesis that $G$ is abelian.

20. Suppose $G$ is a group such that $x^2 = e$ for all $x \in G$. Then since $x \cdot x^{-1} = e$ for all $x \in G$, left cancellation (Proposition 3.1.6) implies that $x = x^{-1}$ for all $x \in G$.

Now let $a, b \in G$. Then by Proposition 3.1.2 part (c), $(ab)^{-1} = b^{-1}a^{-1}$. Using what we have already shown, $(ab)^{-1} = ab$, $b^{-1} = b$ and $a^{-1} = a$. Thus, $ab = ba$ and $G$ is abelian.

6. Let $S$ be a set, $a \in S$ and $H = \{ \sigma \in \text{Sym}(S) : \sigma(a) = a \}$. Clearly $H$ is not empty since the identity permutation is in $H$. Suppose $\sigma, \tau \in H$. Then $\tau^{-1}(a) = \tau^{-1}(\tau(a)) = a$, and so $\sigma(\tau^{-1}(a)) = \sigma(a) = a$. Thus $\sigma \circ \tau^{-1} \in H$. By Corollary 3.2.3, $H$ is a subgroup.

8. Let $G$ be an abelian group and let $H = \{ a \in G : 2a = 0 \}$. Certainly $H$ is not empty since $0 = 0 + 0$, so $0 \in H$. Suppose $a, b \in H$. Then $2(a - b) = 2a - 2b = 0 - 0 = 0$, so $a - b \in H$ and $H$ is a subgroup by Corollary 3.2.3. (When using additive notation, “$-b$” is used to denote the inverse of the element $b$.)

If the notation is a stumbling block for you, feel free to use the generic multiplicative notation. Then $H$ is the subset of $a \in G$ with $a^2 = e$. Suppose $a, b \in H$, i.e., $a^2 = b^2 = e$. Then by what we showed above in #20, $b = b^{-1}$. Thus, since $G$ is abelian, we have $(ab^{-1})^2 = (ab)^2 = a^2b^2 = ee = e$. So $ab^{-1} \in H$ and by Corollary 3.2.3 $H$ is a subgroup of $G$.

In the case where $G = \mathbb{Z}_{12}$, it is easy to check that $H = \{[0]_{12}, [6]_{12}\}$.