Lagrange’s Theorem: If $G$ is a finite group and $H$ is a subgroup of $G$, then the order of $H$ divides the order of $G$. In particular, if $a$ is an element of $G$, then the order of $a$ divides the order of $G$.

Why the “In particular” part? If $a \in G$, then $\langle a \rangle$ is a (cyclic) subgroup of $G$, and its order is the order of $a$.

Suppose $G$ is a group of order $n$ and $d | n$. Then $d$ is possibly the order of some element(s) of $G$. Will there always be an element of order $d$?

Suppose $G$ is a group of order $n$. Then $G$ is cyclic if and only if there is an element $a \in G$ of order $n$. Cyclic groups are always abelian (though not all abelian groups are cyclic), and certainly not all groups are abelian. Thus, not all groups of order $n$ will necessarily have an element of order $n$.

What about other possible orders?

Examples: 1) $G = \mathbb{Z}_{12}$ (cyclic)
2) $G = \mathbb{Z}_{19}$ (cyclic?)
3) $G = S_3$ (definitely not cyclic)
4) $G = S_4$ (definitely not cyclic)
Suppose $G$ is a group of order $n$ and $a \in G$. Then the order of $a$ is some divisor of $n$; call it $d$. We can write $n = md$ for some positive integer $m$. The $n$th power of $a$ is

$$a^n = a^{md} = (a^d)^m = e^m = e.$$ 

Let’s apply this to a familiar situation:

Suppose $a \in \mathbb{Z}_n^\times$. Then $a^{\phi(n)} = 1$. In other words, if $n > 1$ and $a$ in an integer relatively prime to $n$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Even more applications of Lagrange’s Theorem:

If $G$ is a group of order $p$, where $p$ is a prime number, then $G$ is a cyclic group and thus abelian.

If $G$ is a group of order less than 6, then $G$ is abelian.

What if the order of $G$ is 6?

What if the order of $G$ is 8?