GROUPS OF ORDER 4

The goal here is to figure out exactly what all groups of order 2, 3, 4, 5, 6 and 7 are.

We’ve already seen that groups of order 2, 3, 5 and 7 are all cyclic, since 2, 3, 5 and 7 are prime numbers. Thus, we know all about such groups; they are essentially the integers modulo \( n \) for \( n = 2, 3, 5 \) and 7. For example, if \( G \) is a group of order 3, it is cyclic; write \( G = \langle g \rangle = \{ e = g^0, g^1, g^2 \} \). Then the multiplication table for \( G \) looks exactly like the addition table for the integers modulo 3.

\[
\begin{array}{c|ccc}
\times & g^0 & g^1 & g^2 \\
g^0 & g^0 & g^1 & g^2 \\
g^1 & g^1 & g^2 & g^0 \\
g^2 & g^2 & g^0 & g^1 \\
\end{array}
\]

That leaves us with groups of order 4 and 6. We saw last week that a group of order 4 must be abelian, but let’s go over it again (this time with a little more experience under our belts).

Suppose \( G \) is a group of order 4. Then by Lagrange’s theorem, an element of \( G \) has order 1, 2 or 4. Of course, the only element with order 1 is the identity. If there is an element of order 4, then \( G \) is cyclic and essentially the integers modulo 4 (its multiplication table will look exactly like the addition table for the integers modulo 4).

So let’s suppose for that no element of \( G \) has order 4. Then all elements have order 1 or 2. In other words, \( x^2 = e \) for all \( x \in G \). By a previous exercise, \( G \) is abelian. But we want to say more. Write \( G = \{ e, a, b, ab \} \). The multiplication table for \( G \) must be (since \( G \) is abelian) as follows:

\[
\begin{array}{c|cccc}
\times & e & a & b & ab \\
e & e & a & b & ab \\
a & a & e & ab & b \\
b & b & ab & e & a \\
ab & ab & b & a & e \\
\end{array}
\]

Is this table familiar?