CAN TWO GROUPS BE THE SAME WITHOUT BEING EQUAL?

On Wednesday we found that a group of order 6 which is not abelian must have an element of order 2 (call it \( f \)) and an element of order 3 (call it \( g \)) with \( fg \neq gf \). The multiplication table is

\[
\begin{array}{cccccc}
\times & e & f & g & g^2 & fg \\
\hline
 e & e & f & g & g^2 & fg \\
f & f & e & fg & g & g^2 \\
g & g & gf & g^2 & e & fg \\
g^2 & g^2 & fg & e & g & gf \\
fg & fg & g^2 & gf & f & e \\
gf & gf & g & fg & g^2 & e \\
\end{array}
\]

Note how \( f, fg \) and \( gf \) are all order 2, \( g \) and \( g^2 \) are order 3, and no element of order 2 commutes with an element of order 3.

We can apply this to the group \( S_3 \). Here we of course have \( e = (1) \), but past that we have choices. For instance, we could use \( f = (1, 2) \) and \( g = (1, 2, 3) \). We could also have \( f = (1, 3) \) and \( g = (1, 2, 3) \). Really the only restriction is that \( f \) must be one of the elements of order 2 and \( g \) must be one of the elements of order 3.

Clearly the \( e, f, g, \ldots \) are just names we chose for these elements. After all, “\( f \)” is a lot shorter (and fits in a table better) than “the function from the set consisting of the numbers 1, 2 and 3 to this set which sends 1 to 2, 2 to 1, and 3 to 3.”

The invertible \( 2 \times 2 \) matrices with entries in the integers modulo 2 are:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\]

As usual, the first matrix here is the identity. The second (as you can check) has order 2 and the third has order 3. They don’t commute. So this group has the exact same multiplication table! We could, for example, set

\[
f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}
\]

So, while these are different groups, they really are the same. The only real difference is the names we choose for the elements.