

## Derivatives Part III: The Chain Rule

### Math 232 Section 2

The chain rule is the “rule” that tells us how the derivative of a composition is related to the derivatives of the separate functions. Suppose  $\mathbf{g}$  is a function from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  and  $\mathbf{f}$  is a function from  $\mathbb{R}^n$  to  $\mathbb{R}^p$ . We can then compose them to get a function  $\mathbf{f} \circ \mathbf{g}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^p$ . What is the total derivative of this composition?

The answer to this question lies in how linear functions are composed. Suppose  $\mathbf{L}_1$  is a linear function from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  and  $\mathbf{L}_2$  is a linear function from  $\mathbb{R}^n$  to  $\mathbb{R}^p$ . Let's write

$$\mathbf{L}_1(\mathbf{x}) = A\mathbf{x} + \mathbf{a}, \quad \mathbf{L}_2(\mathbf{y}) = B\mathbf{y} + \mathbf{b}.$$

Here  $A$  is an  $n \times m$  matrix and  $B$  is a  $p \times n$  matrix,  $\mathbf{a}$  is a vector in  $\mathbb{R}^n$  and  $\mathbf{b}$  is in  $\mathbb{R}^p$ . See what happens when we compose them:

$$\begin{aligned} \mathbf{L}_2(\mathbf{L}_1(\mathbf{x})) &= \mathbf{L}_2(A\mathbf{x} + \mathbf{a}) \\ &= B(A\mathbf{x} + \mathbf{a}) + \mathbf{b} \\ &= BA\mathbf{x} + B\mathbf{a} + \mathbf{b}. \end{aligned}$$

Note that the coefficient matrix for the composition is just the product of the two coefficient matrices  $A$  and  $B$ !

The chain rule, simply put, is that when you compose two differentiable functions, the total derivative (i.e., the matrix of partial derivatives) of the composition is just the product of the two total derivatives of the functions you're composing. Why? If each of the two functions can be approximated by linear functions, then their composition is approximated by the composition of these two linear functions. Further, the coefficient matrix of this composition of linear functions is the product of the two coefficient matrices. That's it; no fancy formulas to remember here, just multiply the two matrices of partial derivatives.

**Example 7:** Let  $f$  be the real-valued function of two variables

$$z = f\left(\begin{matrix} x \\ y \end{matrix}\right) = xe^y + ye^{-x}.$$

Then  $f$  has a  $1 \times 2$  matrix of partial derivatives

$$\left(\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}\right) = (e^y - ye^{-x} \quad xe^y + e^{-x})$$

Let  $\mathbf{g}$  be the function from  $\mathbb{R}^2$  to  $\mathbb{R}^2$

$$\mathbf{g} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} e^t \\ st^2 \end{pmatrix}.$$

Think of  $\mathbf{g}$  as giving us  $x$  and  $y$ . Then  $\mathbf{g}$  has a  $2 \times 2$  matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & e^t \\ t^2 & 2st \end{pmatrix}$$

The composition  $z = f \circ \mathbf{g}$  is a real-valued function of two variables,  $s$  and  $t$ , with a  $1 \times 2$  matrix of partial derivatives

$$\begin{aligned} \left( \frac{\partial z}{\partial s} \quad \frac{\partial z}{\partial t} \right) &= \left( \frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y} \right) \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \\ &= (e^y - ye^{-x} \quad xe^y + e^{-x}) \begin{pmatrix} 0 & e^t \\ t^2 & 2st \end{pmatrix} \\ &= ((xe^y + e^{-x})t^2 \quad (e^y - ye^{-x})e^t + (xe^y + e^{-x})2st). \end{aligned}$$

Another way to express this (without matrices) is with the two equations

$$\begin{aligned} \frac{\partial z}{\partial s} &= (xe^y + e^{-x})t^2 \\ \frac{\partial z}{\partial t} &= (e^y - ye^{-x})e^t + (xe^y + e^{-x})2st. \end{aligned}$$

In particular, we can find the total derivative of  $z$  at the point  $s = 1$ ,  $t = 0$  by first finding what  $x$  and  $y$  are, and then plugging these values into the “formula” we found. When  $s = 1$  and  $t = 0$ ,

$$x = e^t = e^0 = 1, \quad y = st^2 = 0^2 = 0.$$

So

$$\begin{aligned} \frac{\partial z}{\partial s}(1, 0) &= (1e^0 + e^{-1})0^2 = 0 \\ \frac{\partial z}{\partial t}(1, 0) &= (e^0 - 0e^{-1})e^0 + (e^0 + e^{-1})0 = 1. \end{aligned}$$

**Exercises 12-19.** Do the assigned exercises from section 15.5 of the textbook, except delete number 17 and their phony-bologna “tree diagram”.