1. Find the velocity, speed and acceleration of the particle with position given by \( \mathbf{r}(t) = t^2 \mathbf{i} - \sin t \mathbf{j} + \tan^{-1} t \mathbf{k} \).

The velocity is the derivative of position,

\[
\mathbf{r}'(t) = 2t \mathbf{i} - \cos t \mathbf{j} + \frac{1}{1 + t^2} \mathbf{k}.
\]

The speed is the magnitude of velocity,

\[
|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (\cos t)^2 + (1 + t^2)^{-2}}.
\]

The acceleration is the derivative of velocity,

\[
\mathbf{r}''(t) = 2 \mathbf{i} + \sin t \mathbf{j} - 2t(1 + t^2)^{-2} \mathbf{k}.
\]

2. Let \( w = xyz - \sin(x + z) + \cos y \), where \( x = s \cos t \), \( y = \sin t + s \) and \( z = e^{st} \). Use the chain rule to find \( \partial w/\partial s \) and \( \partial w/\partial t \).

Find the two total derivatives (matrices of partial derivatives) first. One is

\[
\left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) = \left( \begin{array}{ccc}
yz - \cos(x + z) & xz - \sin y & xy - \cos(x + z) \\
\end{array} \right)
\]

and the other is

\[
\left( \begin{array}{ccc}
\cos t & -s \sin t & 1 \\
-\sin t & \cos t & 0 \\
te^{st} & 0 & se^{st} \\
\end{array} \right).
\]

By the chain rule, \( \left( \frac{\partial w}{\partial s} \frac{\partial w}{\partial t} \right) \) is the product of these two matrices, which is

\[
\left( \begin{array}{ccc}
yz - \cos(x + z) & xz - \sin y & xy - \cos(x + z) \\
\end{array} \right) \left( \begin{array}{ccc}
\cos t & -s \sin t & 1 \\
-\sin t & \cos t & 0 \\
te^{st} & 0 & se^{st} \\
\end{array} \right).
\]

3. Evaluate the following limits.
a) \[ \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2 + y^2 + z^2}{x^2 - y^2 - z^2} \] 

Convert to spherical coordinates. The numerator is just \( \rho^2 \). There is also a \( \rho^2 \) factor in the denominator, \( \rho^2 (\cos^2 \theta \sin^2 \phi - \sin^2 \theta \sin^2 \phi - \cos^2 \phi) \). The limit is equal to

\[
\lim_{\rho \rightarrow 0^+} \frac{\rho^2}{\rho^2 (\cos^2 \theta \sin^2 \phi - \sin^2 \theta \sin^2 \phi - \cos^2 \phi)} = \lim_{\rho \rightarrow 0^+} \frac{1}{\cos^2 \theta \sin^2 \phi - \sin^2 \theta \sin^2 \phi - \cos^2 \phi},
\]

which clearly doesn’t exist since it depends on the angles \( \theta \) and \( \phi \).

b) \[ \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y - xy^2}{x^2 + y^2} \]

Convert to polar coordinates. This limit is

\[
\lim_{r \rightarrow 0^+} \frac{r^3 \cos^2 \theta \sin \theta - r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0^+} r (\cos^2 \theta \sin \theta - \cos \theta \sin^2 \theta) = 0.
\]

4. Let \( f(x, y) = xye^x \). Find the directional derivative of \( f \) at the point \( (1, 2) \) in the direction of \( (2, 1) \).

The directional derivative is the dot product of the gradient with the direction. First,

\[
\nabla f = \langle f_x, f_y \rangle = \langle ye^x + xye^x, xe^x \rangle.
\]

So

\[
\nabla f(1, 2) = \langle 4e, e \rangle.
\]

Next, the direction of \( \langle 1, 2 \rangle \) is \( \sqrt{5}^{-1} \langle 1, 2 \rangle \). So the directional derivative is \( \sqrt{5}^{-1} 6e \).

5. Find the absolute maximum and minimum values of \( x^2 - 2xy + y \) on the rectangle \( R = \{(x, y): -1 \leq x \leq 1, -2 \leq y \leq 2 \} \).

The first step is to find the critical points inside the rectangle. To do that, you solve \( f_x = f_y = 0 \). In this case, we get \( 2x - 2y = -2x + 1 = 0 \), and the only solution is \( (1/2, 1/2) \) (which is in the rectangle). At this point, the function value is \( 1/4 \).

Next, we parametrize the boundary of the rectangle. This is best done in four pieces. The top of the rectangle is given by

\[
x = t, \quad y = 2, \quad -1 \leq t \leq 1.
\]
The right hand side is given by
\[ x = 1, \ y = t, \ -2 \leq t \leq 2. \]

The bottom is given by
\[ x = t, \ y = -2, \ -1 \leq t \leq 1. \]

The left hand side is given by
\[ x = -1, \ y = t, \ -2 \leq t \leq 2. \]

On the top of the rectangle, the function \( f \) is given by
\[ f(t, 2) = t^2 - 4t + 2, \ -1 \leq t \leq 1, \]
which (as you can check) has an absolute maximum of \( f(-1, 2) = 7 \) and an absolute minimum of \( f(1, 2) = -1 \).

On the right hand side of the rectangle, the function \( f \) is given by
\[ f(1, t) = 1 - t, \ -2 \leq t \leq 2, \]
which has an absolute maximum of \( f(1, -2) = 3 \) and an absolute minimum of \( f(1, 2) = -1 \).

On the bottom of the rectangle, the function \( f \) is given by
\[ f(t, -2) = t^2 + 4t - 2, \ -1 \leq t \leq 1, \]
which has an absolute maximum of \( f(1, -2) = 3 \) and an absolute minimum of \( f(-1, -2) = -5 \).

Finally, on the left hand side of the rectangle, the function \( f \) is given by
\[ f(-1, t) = 1 + 3t, \ -2 \leq t \leq 2, \]
which has an absolute maximum of \( f(-1, 2) = 7 \) and an absolute minimum of \( f(-1, -2) = -5 \).

The absolute maximum of \( f \) on the rectangle is \( f(-1, 2) = 7 \) and the absolute minimum is \( f(-1, -2) = -5 \).

6. Find the linearization of \( f(x, y) = e^{x+y} - \ln(x^2 + y^2) \) at the point \((0,1)\) and use this to approximate \( f(.1, .9) \).
The linearization of $f$ at $(0, 1)$ is

$$L(x, y) = f_x(0, 1)(x - 0) + f_y(y - 1) + f(0, 1).$$

The partial derivatives are

$$f_x = e^{x+y} - 2x/(x^2 + y^2) \quad \text{and} \quad f_y = e^{x+y} - 2y/(x^2 + y^2).$$

The function value is

$$f(0, 1) = e.$$

Plugging $x = 0$ and $y = 1$ into the partial derivatives, you get

$$L(x, y) = e(x - 0) + (e - 2)(y - 1) + e.$$

Finally, $f(.1, .9)$ is approximately

$$L(.1, .9) = e(.1) + (e - 2)(-.1) + e.$$