Math 581 Spring 2006
Homework #1
Due January 27, 2006

Let $\omega = \frac{1+\sqrt{-3}}{2}$ and consider the ring

$$\mathbb{Z}[\omega] = \{a + b\omega\colon a, b \in \mathbb{Z}\}.$$

Let $|\cdot|$ denote the usual complex modulus:

$$|a + b\omega| = \sqrt{(a + b\omega)(a + b\overline{\omega})},$$

where $\overline{\omega} = \frac{1-\sqrt{-3}}{2}$ (the complex conjugate of ω).

Fix an $a + b\omega \in \mathbb{Z}[\omega]$ and view the complex numbers $\mathbb{C}$ as $\mathbb{R}^2$ in the usual way. As we did in class, show that every complex number α is in a quadrilateral whose vertices are multiples (in $\mathbb{Z}[\omega]$) of $a + b\omega$. What sort of quadrilaterals are these? What are the dimensions?

Find a positive real number c so that the following statement is true: given any complex number α, there is a $q \in \mathbb{Z}[\omega]$ such that

$$\alpha = q(a + b\omega) + r, \qquad |r| \leq c|a + b\omega|.$$

Show that $\mathbb{Z}[\omega]$ is a principal ideal domain.

Show that an element $u \in \mathbb{Z}[\omega]$ is invertible if and only if $|u| = 1$. Find all such elements u. In other words, find all the units of $\mathbb{Z}[\omega]$.

Show that $\mathbb{Z}[\omega]$ is a unique factorization domain.

In general, for a square-free integer $D \equiv 2, 3 \mod 4$, let $\mathfrak{O}(D) = \mathbb{Z}[\sqrt{D}]$, and if $D \equiv 1 \mod 4$, let $\mathfrak{O}(D) = \mathbb{Z}[(1 + \sqrt{D})/2]$. Between the above and what we did in class, you've seen the cases $D = -1, -2, -3$ and -5. Use any resources at your disposal (literature search, google, wikipedia, usenet, bribery, etc...) to find the answer to the following question: For which square-free $D < 0$ is the ring $\mathfrak{O}(D)$ a principal ideal domain?