Math 581 Spring 2006
Homework for Week #12
Due April 21, 2006

In what follows, m denotes a negative square-free rational integer, $K = \mathbb{Q}(\sqrt{m})$ and $\mathfrak{O}_K$ denotes (as usual) the ring of integers in K.

You may assume that $\mathfrak{O}_K$ is a Euclidean domain when $m = -1, -2, -3$, since those cases were dealt with either in class or previous homework.

1. Prove that $\mathfrak{O}_K$ is a Euclidean domain when $m = -7, -11$, but is not a Euclidean domain when $m = -5, -10$.

2. Find all units in $\mathfrak{O}_K$, assuming that $m < -11$.

3. Assume that $\mathfrak{O}_K$ is a Euclidean domain and let $\alpha \in \mathfrak{O}_K \setminus \{0\}$ be a non-unit with minimal size (here "size" refers to the function $f\colon \mathfrak{O}_K \to \mathbb{N}$ which makes $\mathfrak{O}_K$ a Euclidean domain). Show that $[\mathfrak{O}_K\colon (\alpha)]$ is either 2 or 3.

4. Assuming that $m < -11$, show that $\mathfrak{O}_K$ contains no α with $N\big((\alpha)\big)$ equal to 2 or 3. Conclude that $\mathfrak{O}_K$ is a Euclidean domain precisely when $m = -1, -2, -3, -7, -11$.

5. Give four examples of a principal ideal domain which is not a Euclidean domain, with proof.