Math 581 Spring 2006
Homework #2
Due February 10, 2006

1. In class we showed how to find a monic polynomial with integer coefficients that has $\sqrt{2}+\sqrt{3}$ as a root, so that $\sqrt{2}+\sqrt{3}$ is an algebraic integer. Use this technique to show that $\sqrt[3]{2}+\sqrt{2}$ is an algebraic integer.

2. Let K be a number field with ring of integers $\mathfrak{O}_K$. We proved that if $\beta \in \mathfrak{O}_K$, then $N_{K/\mathbb{Q}}(\beta) \in \mathbb{Z}$. The converse is not true, however. Find an example of a K and a $\beta \in K$ with $N_{K/\mathbb{Q}}(\beta) \in \mathbb{Z}$ but $\beta \notin \mathfrak{O}_K$.

3. Let K be a number field with ring of integers $\mathfrak{O}_K$ and let $\mathfrak{P}$ be a non-zero prime ideal of $\mathfrak{O}_K$. Prove that $\mathfrak{P}$ contains a unique prime number $p \in \mathbb{Z}$. Conclude that $N(\mathfrak{P}) = p^f$ for some positive $f \in \mathbb{Z}$.

4. Let K be a number field with ring of integers $\mathfrak{O}_K$ and let β be a non-zero element of $\mathfrak{O}_K$. Prove that $N\big((\beta)\big) = |N_{K/\mathbb{Q}}(\beta)|$, where (β) denotes the principal ideal of $\mathfrak{O}_K$ generated by β.

5. Find an example of a number field K and an ideal $\mathfrak{A} \subset \mathfrak{O}_K$ which is not principal.