Math 581 Spring 2006
Homework #3
Due February 24, 2006

To set notation, K will denote a number field of degree n over $\mathbb{Q}$ with ring of integers $\mathfrak{O}_K$ and discriminant Δ_K. Let $\sigma_1, \ldots, \sigma_n$ denote the embeddings of K into $\mathbb{C}$. Given any n elements $\alpha_1, \ldots, \alpha_n \in K$, we form the $n \times n$ matrix

$$\begin{pmatrix} \sigma_1(\alpha_1) & \ldots & \sigma_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_n(\alpha_1) & \ldots & \sigma_n(\alpha_n) \end{pmatrix}$$

and denote the square of its determinant by $\operatorname{disc}(\alpha_1, \ldots, \alpha_n)$. In the case where the alphas are a $\mathbb{Z}$-basis for $\mathfrak{O}_K$, by definition $\operatorname{disc}(\alpha_1, \ldots, \alpha_n) = \Delta_K$. We showed in class that $\operatorname{disc}(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}$ whenever all the alphas are in $\mathfrak{O}_K$.

For a singleton $\alpha \in K$, define

$$\operatorname{disc}(\alpha) := \operatorname{disc}(1, \alpha, \ldots, \alpha^{n-1}).$$

You may assume #27c on page 45 of the textbook for these exercises.

1. Show that $\Delta_K \equiv 0$ or $1 \mod 4$. This is called Stickelberger's criterion. See exercise 22 on page 43 of the textbook.

2. Suppose $\alpha_1, \ldots, \alpha_n \in \mathfrak{O}_K$ are linearly independent over $\mathbb{Q}$ and $\operatorname{disc}(\alpha_1, \ldots, \alpha_n)$ is a square-free rational integer. Prove that the alphas are a $\mathbb{Z}$-basis for $\mathfrak{O}_K$, i.e., that $\operatorname{disc}(\alpha_1, \ldots, \alpha_n) = \Delta_K$.

3. Do exericse 28 on page 46 of the textbook.

4. Let α be a root of $X^3 - X - 1$ and let $K = \mathbb{Q}(\alpha)$ (see #3 above). Find the ramification indices and residue class (inertial) degrees of the prime ideals of $\mathfrak{O}_K$ which lie above 2, 5 and 23. For those with a tiny bit of computer savvy, can you find a prime which splits completely?

5. Let $\mathfrak{A}$ be a non-zero fractional ideal. Prove that

$$\mathfrak{A} = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$$

for some $\alpha_1, \ldots, \alpha_n \in K$ and that $\operatorname{disc}(\alpha_1, \ldots, \alpha_n) = \Delta_K N(\mathfrak{A})^2$. Hint: remember that there is a non-zero integer α such that $(\alpha)\mathfrak{A}$ is a non-zero ideal, and that we've already proven this if $\mathfrak{A}$ is an ideal.

Let $\mathfrak{B}$ be non-zero fractional ideal such that $\mathfrak{A} \supseteq \mathfrak{B}$, i.e., $\mathfrak{B} = \mathfrak{I}\mathfrak{A}$ where $\mathfrak{I}$ is a non-zero ideal. Thus, we can view $\mathfrak{B}$ as a subring of $\mathfrak{A}$. Prove that

$$[\mathfrak{A} \colon \mathfrak{B}] = \frac{N(\mathfrak{B})}{N(\mathfrak{A})},$$

i.e., $[\mathfrak{A} \colon \mathfrak{B}] = N(\mathfrak{I})$. (Cf. exercise 32 on page 92 of the textbook.)