A Useful Lemma Regarding Units

We fix a number field K. The degree of K over $\mathbb{Q}$ is denoted by n. There are n embeddings $\sigma\colon K \to \mathbb{C}$; there are r embeddings into $\mathbb{R}$ and s pairs of complex conjugate embeddings into $\mathbb{C}$ (not real). Thus $n = r + 2s$. These embeddings are ordered so that $\sigma_i\colon K \to \mathbb{R}$ for $i \le r$ and $\sigma_{i+s} = \overline{\sigma_i}$ for $r+1 \le i \le r+s$, where the overline denotes complex conjugation. As usual $\sqrt{|\Delta_K|}$, denotes the square root of the absolute value of the discriminant of K. We also use

$$e_i = \begin{cases} 1 & \text{if } i \le r, \\ 2 & \text{if } r+1 \le i \le r+s. \end{cases}$$

Define $\rho\colon K \to \mathbb{R}^n$ by

$$\rho(\alpha) = \Big(\sigma_1(\alpha), \dots, \sigma_r(\alpha), \Re\big(\sigma_{r+1}(\alpha)\big), \dots, \Re\big(\sigma_{r+s}(\alpha)\big), \Im\big(\sigma_{r+1}(\alpha)\big), \dots, \Im\big(\sigma_{r+s}(\alpha)\big)\Big).$$

For positive real numbers $a_1, \dots, a_{r+s}$, define

$$C(a_1, \dots, a_{r+s}) := \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \colon |x_i| \le a_i,\ 1 \le i \le r \text{ and } x_i^2 + x_{i+s}^2 \le a_i^2,\ r < i \le r_s\}.$$

It is not difficult to see that such a set is a convex body in $\mathbb{R}^n$ with

$$\operatorname{Vol}\big(C(a_1, \dots, a_{r+s})\big) = 2^r \pi^s \prod_{i=1}^{r+s} a_i^{e_i}.$$

Lemma: Suppose $r + s > 1$ and let $i_0 \in \{1, \dots, r+s\}$. There are infinitely many non-zero $\alpha \in \mathfrak{O}_K$ with $|N_{K/\mathbb{Q}}(\alpha)| \le \sqrt{|\Delta_K|}(2/\pi)^s$ and $|\sigma_i(\alpha)| < 1$ for all $i \ne i_0$. There is a unit $u \in \mathfrak{O}_K^\times$ with $|\sigma(u)| < 1$ for all $i \ne i_0$.

Proof: Let $a_i = 1/2$ for $i \ne i_0$ and let a_{i_0} be the positive real number satisfying $\prod_{i=1}^{r+s} a_i^{e_i} = \sqrt{|\Delta_K|}(2/\pi)^s$. Since $\det\big(\rho(\mathfrak{O}_K)\big) = 2^{-s}\sqrt{|\Delta_K|}$, by Minkowski's theorem there is a non-zero $\alpha_1 \in \mathfrak{O}_K$ with $\rho(\alpha_1) \in C(a_1, \dots, a_{r+s})$. By the definition of $C(a_1, \dots, a_{r+s})$ and ρ, we have $|\sigma_i(\alpha_1)| \le a_i$ for $i = 1, \dots, r+s$. Thus,

$$|N_{K/\mathbb{Q}}(\alpha_1)| \le \prod_{i=1}^{r+s} a_i^{e_i} = \sqrt{|\Delta_K|}(2/\pi)^s$$

and $|\sigma_i(\alpha_1)| \le 1/2$ for all $i \ne i_0$.

Now let $a_i = \frac{|\sigma(\alpha_1)|}{2}$ for $i \ne i_0$. This will yield a non-zero α_2 satisfying the statement of the lemma and also $|\sigma_i(\alpha_2)| \le |\sigma_i(\alpha_1)|/2$ for all $i \ne i_0$. Continue on in this fashion, getting a sequence $\alpha_1, \alpha_2, \dots$ of non-zero integers which satisfy the statement of the lemma and also

$$|\sigma_i(\alpha_1)| > |\sigma_i(\alpha_2)| > \cdots$$

for all $i \ne i_0$.

Obviously these α_j's are distinct. But there are only finitely many integral ideals with norm no greater than $\sqrt{|\Delta_K|}(2/\pi)^s$. Hence, the principal ideals (α_j) cannot all be distinct; $(\alpha_l) = (\alpha_m)$ for some indices $l < m$. This forces $\alpha_m = u\alpha_l$ for some unit $u \in \mathfrak{O}_K^\times$. Further,

$$|\sigma_i(u)||\sigma_i(\alpha_l)| = |\sigma_i(\alpha_m)| < |\sigma_i(\alpha_l)|$$

for all $i \ne i_0$. This completes the proof.