A Useful Lemma Regarding Units

We fix a number field \( K \). The degree of \( K \) over \( \mathbb{Q} \) is denoted by \( n \). There are \( n \) embeddings \( \sigma: K \to \mathbb{C} \); there are \( r \) embeddings into \( \mathbb{R} \) and \( s \) pairs of complex conjugate embeddings into \( \mathbb{C} \) (not real). Thus \( n = r + 2s \). These embeddings are ordered so that \( \sigma_i: K \to \mathbb{R} \) for \( i \leq r \) and \( \sigma_{i+s} = \overline{\sigma}_i \) for \( r + 1 \leq i \leq r + s \), where the overline denotes complex conjugation. As usual \( \sqrt{|\Delta_K|} \), denotes the square root of the absolute value of the discriminant of \( K \). We also use

\[
e_i = \begin{cases} 1 & \text{if } i \leq r, \\ 2 & \text{if } r + 1 \leq i \leq r + s. \end{cases}
\]

Define \( \rho: K \to \mathbb{R}^n \) by

\[
\rho(\alpha) = \left( \sigma_1(\alpha), \ldots, \sigma_r(\alpha), \Re(\sigma_{r+1}(\alpha)), \ldots, \Re(\sigma_{r+s}(\alpha)), \Im(\sigma_{r+1}(\alpha)), \ldots, \Im(\sigma_{r+s}(\alpha)) \right).
\]

For positive real numbers \( a_1, \ldots, a_{r+s} \), define

\[
C(a_1, \ldots, a_{r+s}) := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq a_i, 1 \leq i \leq r \text{ and } x_i^2 + x_{i+s}^2 \leq a_i^2, r < i \leq r+s \}.
\]

It is not difficult to see that such a set is a convex body in \( \mathbb{R}^n \) with

\[
\Vol(C(a_1, \ldots, a_{r+s})) = 2^r \pi^s \prod_{i=1}^{r+s} a_i^{e_i}.
\]

Lemma: Suppose \( r + s > 1 \) and let \( i_0 \in \{1, \ldots, r+s\} \). There are infinitely many non-zero \( \alpha \in \mathfrak{O}_K \) with \( |N_{K/\mathbb{Q}}(\alpha)| \leq \sqrt{|\Delta_K|}(2/\pi)^s \) and \( |\sigma_i(\alpha)| < 1 \) for all \( i \neq i_0 \). There is a unit \( u \in \mathfrak{O}_K^\times \) with \( |\sigma(u)| < 1 \) for all \( i \neq i_0 \).

Proof: Let \( a_i = 1/2 \) for \( i \neq i_0 \) and let \( a_{i_0} \) be the positive real number satisfying \( \prod_{i=1}^{r+s} a_i^{e_i} = \sqrt{|\Delta_K|}(2/\pi)^s \). Since \( \det(\rho(\mathfrak{O}_K)) = 2^{-s} \sqrt{|\Delta_K|} \), by Minkowski’s theorem there is a non-zero \( \alpha_1 \in \mathfrak{O}_K \) with \( \rho(\alpha_1) \in C(a_1, \ldots, a_{r+s}) \). By the definition of \( C(a_1, \ldots, a_{r+s}) \) and \( \rho \), we have \( |\sigma_i(\alpha_1)| \leq a_i \) for \( i = 1, \ldots, r+s \). Thus,

\[
|N_{K/\mathbb{Q}}(\alpha_1)| \leq \prod_{i=1}^{r+s} a_i^{e_i} = \sqrt{|\Delta_K|}(2/\pi)^s
\]

and \( |\sigma_i(\alpha_1)| \leq 1/2 \) for all \( i \neq i_0 \).

Now let \( a_i = |\sigma_i(\alpha_1)/2| \) for \( i \neq i_0 \). This will yield a non-zero \( \alpha_2 \) satisfying the statement of the lemma and also \( |\sigma_i(\alpha_2)| \leq |\sigma_i(\alpha_1)|/2 \) for all \( i \neq i_0 \). Continue on in this fashion, getting a sequence \( \alpha_1, \alpha_2, \ldots \) of non-zero integers which satisfy the statement of the lemma and also

\[
|\sigma_i(\alpha_1)| > |\sigma_i(\alpha_2)| > \cdots
\]

for all \( i \neq i_0 \).

Obviously these \( \alpha_j \)'s are distinct. But there are only finitely many integral ideals with norm no greater than \( \sqrt{|\Delta_K|}(2/\pi)^s \). Hence, the principal ideals \( (\alpha_j) \) cannot all be distinct; \( (\alpha_i) = (\alpha_m) \) for some indices \( i < m \). This forces \( \alpha_m = u\alpha_i \) for some unit \( u \in \mathfrak{O}_K^\times \). Further,

\[
|\sigma_i(u)||\sigma_i(\alpha_i)| = |\sigma_i(\alpha_m)| < |\sigma_i(\alpha_i)|
\]

for all \( i \neq i_0 \). This completes the proof.