# Dedekind's Zeta Function

We fix a number field $K$ with ring of integers $\mathfrak{O}_K$. The degree of $K$ over $\mathbb{Q}$ is denoted by $d$. As usual, $h, R, w, \Delta_K$ denote the class number, regulator, number of roots of unity, and discriminant of $K$, respectively. Further, we denote the number of real embeddings of $K$ into $\mathbb{C}$ by $r$ and use $s$ for the number of pairs of complex conjugate embeddings.

For a positive integer $n$, let $a_n$ be the number of integral ideals with norm $n$. The zeta function of $K$ is

$$\zeta_K(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for real $s > 1$. It's traditional to drop the subscript when $K = \mathbb{Q}$. It's somewhat traditional to let

$$\chi := \frac{R}{w\sqrt{|\Delta_K|}} 2^{r+2}\pi^s.$$

**Theorem**: The series for $\zeta_K$ converges (absolutely) for all $s > 1$. Moreover,

$$\lim_{s \to 1^+} (s-1)\zeta_K(s) = h\chi.$$

Note that we already proved this for $K = \mathbb{Q}$.

**Proof**: Write $S(n) = a_1 + \cdots + a_n$ for $n \geq 1$ and set $S(0) = 0$. Then by the Dedekind-Weber Theorem

$$S(n) = h\chi n + O(n^{1-(1/d)}), \tag{1}$$

where the implicit constant depends only on $K$. In particular, we have

$$\frac{S(n)}{n} < C \qquad n \geq 1$$

for some $C > 0$. Using this, we have for any $m \geq 1$

$$\begin{aligned}
\sum_{n=1}^{m} \frac{a(n)}{n^s} &= \sum_{n=1}^{m} \frac{S(n) - S(n-1)}{n^s} \\
&= \frac{S(m)}{m^s} + \sum_{n=1}^{m-1} S(n) \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\
&= \frac{S(m)}{m^s} + \sum_{n=1}^{m-1} S(n) s \int_n^{n+1} x^{-(s+1)}\, dx \\
&< \frac{C}{m^{s-1}} + \sum_{n=1}^{m-1} \frac{S(n)}{n} s \int_n^{n+1} x^{-s}\, dx \\
&< \frac{C}{m^{s-1}} + Cs \int_1^{m} x^{-s}\, dx \\
&< \frac{C}{m^{s-1}} + Cs \int_1^{\infty} x^{-s}\, dx \\
&= \frac{C}{m^{s-1}} + \frac{Cs}{s-1}.
\end{aligned}$$

This shows that the series converges absolutely.

Now write

$$S(n) = h\chi n + \epsilon(n).$$

Then

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{S(n) - S(n-1)}{n^s} = \sum_{n=1}^{\infty} \frac{h\chi}{n^s} + \frac{\epsilon(n) - \epsilon(n-1)}{n^s}.$$

Using this,

$$\begin{aligned} |\zeta_K(s) - h\chi\zeta(s)| &= \left| \sum_{n=1}^{\infty} \frac{\epsilon(n) - \epsilon(n-1)}{n^s} \right| \\ &= \left| \sum_{n=1}^{\infty} \epsilon(n) \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right| \\ &= \left| \sum_{n=1}^{\infty} s\epsilon(n) \int_n^{n+1} x^{-(s+1)}\, dx \right| \\ &\leq s \sum_{n=1}^{\infty} |\epsilon(n)| \int_n^{n+1} x^{-(s+1)}\, dx. \end{aligned}$$

Now by (1), we have

$$|\epsilon(n)| < C' n^{1-(1/d)} \qquad n \geq 1$$

for some $C' > 0$ depending only on $K$. Using this with the above gives

$$\begin{aligned} |\zeta_K(s) - h\chi\zeta(s)| &< s \sum_{n=1}^{\infty} C' n^{1-(1/d)} \int_n^{n+1} x^{-(s+1)}\, dx \\ &< C's \sum_{n=1}^{\infty} \int_n^{n+1} x^{-s-(1/d)}\, dx \\ &< C's \int_1^{\infty} x^{-1-(1/d)}\, dx \\ &= C'sd. \end{aligned}$$

We now have

$$|(s-1)\zeta_K(s) - (s-1)h\chi\zeta(s)| < C'sd(s-1)$$

for all $s > 1$. Thus,

$$\lim_{s \to 1^+} (s-1)\zeta_K(s) = \lim_{s \to 1^+} (s-1)h\chi\zeta(s) = h\chi.$$